DS-Weights: An Analytical User Guide*

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In this user guide, we explain how to use Dynamic Stochastic Weights (DS-weights) — introduced in Dávila and Schaab (2022) — in the context of several applications. This guide is a living document, which will be expanded and updated regularly.

This user guide has a dual purpose. Its first purpose is to explain the mechanics of how to define and use DS-weights in specific environments. Its second purpose is to show that, by using DS-weights, it is possible to obtain new economic insights into welfare assessments in a way that was not possible before.

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1 Application #1: Linear Labor Income Taxation with Stochastic Earnings (based on Piketty and Saez (2013))

In their Handbook chapter on Optimal Labor Income Taxation, Piketty and Saez (2013) illustrate how the linear labor income taxation model considerably simplifies the exposition while capturing the key equity-efficiency trade-off that underlies the literature on optimal labor income taxation. This model, typically traced back to Sheshinski (1972), follows the nonlinear income tax analysis of Mirrlees (1971). While our results can also be extended to the more complex case with nonlinear taxes, here we focus on the linear case. See also Saez and Stantcheva (2016) and Kaplow (2007) for more details on this model.

The baseline labor income taxation model assumes that individuals are ex-ante heterogeneous, either on productivity or preferences, but it does not allow for random earnings. This extension is taken up by Varian (1980) and Eaton and Rosen (1980), among others — see Section 9.2 of Kaplow (2007) for a detailed discussion of this body of work. However, Piketty and Saez (2013) write that

> *"Therefore, the random earnings model generates both the same equity-efficiency trade-off and the same type of optimal tax formula."*

In this application, we show that the deterministic earnings model features a tradeoff between Aggregate Efficiency and Redistribution, but that the stochastic earnings model trades off Aggregate Efficiency, Risk-Sharing, and Redistribution, and this may have important consequences for the determination of optimal taxes.

1.1 Deterministic earnings

Environment. We initially consider a single-period environment in which individuals have to a make consumption-labor decision. We assume that individuals have preferences of the form

$$u_i\left(c^i, n^i\right),$$

where c^i denotes consumption and n^i denotes hours worked. We assume that $\frac{\partial u_i\left(c^i,n^i\right)}{\partial c^i} > 0$, $\frac{\partial u_i\left(c^i,n^i\right)}{\partial n^i} < 0$, and any other needed regularity conditions. When individuals face a linear labor income tax, the budget constraint of individual i is given by

$$c^i = (1-\tau)\, w^i n^i + g,$$

where τ is the constant linear tax rate and g is a uniform per-capita grant, i.e., a demogrant.[1]

Individual optimality. The problem that an individual faces can be expressed in terms of the following Lagrangian:

$$\mathcal{L}^i = u_i\left(c^i, n^i\right) - \lambda^i\left(c^i - (1-\tau)\, w^i n^i - g\right),$$

whose solution implies that

$$(1-\tau)\, w^i \frac{\partial u_i\left(c^i, n^i\right)}{\partial c^i} + \frac{\partial u_i\left(c^i, n^i\right)}{\partial n^i} = 0.$$

[1] We could alternatively have written an individual's budget constraint as $c^i = w^i n^i - T\left(w^i n^i\right)$, where $T\left(w^i n^i\right) = \tau w^i n^i - g$.

Hence, we can define an indirect utility function for individual i in terms of τ and g as

$$V_i(\tau, g) = \max u_i\left(c^i, n^i\right) \quad \text{s.t.} \quad c^i = (1-\tau)\, w^i n^i + g.$$

Under the assumption that g can be written as a function of τ, as in $g(\tau)$, we can write the total change in indirect utility for individual i as[2]

$$\frac{dV_i}{d\tau} = \frac{\partial u_i\left(c^i, n^i\right)}{\partial c^i} \underbrace{\left(-w^i n^i + \frac{dg}{d\tau}\right)}_{=\frac{du_{i|c}}{d\tau}},$$

where $\frac{du_{i|c}}{d\tau}$ denotes the consumption-equivalent effect of the policy, which is expressed in consumption units.

Optimal linear income tax. The government chooses τ and g to maximize a particular social welfare function $\mathcal{W}(\cdot)$ subject to a revenue constraint and to the constraints that represent individual behavior. Formally,

$$W(\tau) = \mathcal{W}\left(\{V_i(\tau, g(\tau))\}_{i \in I}\right),$$

where the mapping $g(\tau)$ is defined by the government's budget constraint, which takes the form

$$E = \tau \int w^i n^i\left((1-\tau)\, w^i, g\right) di - g, \tag{1}$$

where the function $n^i\left((1-\tau)\, w^i, g\right)$ denotes individual i's (Marshallian) labor supply. That is, it is evident that from Equation (1) it is possible to express g as a function of τ.

A welfare assessment for a welfarist planner takes the form

$$\begin{aligned} \frac{dW(\tau)}{d\tau} &= \int \frac{\partial \mathcal{W}}{\partial V_i} \frac{dV_i(\tau, g(\tau))}{d\tau} di \\ &= \int \frac{\partial \mathcal{W}}{\partial V_i} \frac{\partial u_i\left(c^i, n^i\right)}{\partial c^i} \left(-w^i n^i + \frac{dg}{d\tau}\right) di. \end{aligned}$$

Since this is a static environment, a normalized DS-Planner will only need to rely on normalized individual weights, which take the form

$$\tilde{\omega}^i = \frac{\frac{\partial \mathcal{W}}{\partial V_i} \frac{\partial u_i\left(c^i, n^i\right)}{\partial c^i}}{\int \frac{\partial \mathcal{W}}{\partial V_i} \frac{\partial u_i\left(c^i, n^i\right)}{\partial c^i} di}.$$

[2]Note that $\frac{dT\left(w^i n^i\right)}{d\tau} = -w^i n^i + \frac{dg}{d\tau}$.

Given these normalized individual weights, we can express the welfare assessments as follows:

$$
\begin{aligned}
\frac{dW^{DS}}{d\tau} &= \int \tilde{\omega}^i \left(-\underbrace{w^i n^i}_{z^i} + \underbrace{\frac{dg}{d\tau}}_{= Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)}} \right) \\
&= \int \tilde{\omega}^i \left(-z^i + Z - \tau \frac{dZ\left(1-\tau\right)}{d\left(1-\tau\right)} \right) \\
&= \int \left(-z^i + Z - \tau \frac{dZ\left(1-\tau\right)}{d\left(1-\tau\right)} \right) di + \mathbb{C}ov_i \left[\tilde{\omega}^i, Z - z^i - \tau \frac{dZ\left(1-\tau\right)}{d\left(1-\tau\right)} \right] \\
&= \underbrace{-\tau \frac{dZ\left(1-\tau\right)}{d\left(1-\tau\right)}}_{=\Xi_{AE} \text{ (Agg. Efficiency)}} + \underbrace{\mathbb{C}ov_i \left[\tilde{\omega}^i, -z^i \right]}_{=\Xi_{RD} \text{ (Redistribution)}} .
\end{aligned} \tag{2}
$$

In the optimal tax literature, it is typical to try to provide an expression for τ in terms of interpretable objects. In fact, solving for τ in Equation (2) yields exactly Equation (3) in Piketty and Saez (2013). We will not insist on doing that there, since our goal is to highlight the use the aggregate additive decomposition introduced Dávila and Schaab (2022). Note that $\Xi_{AE} = 0$ and $\Xi_{RD} > 0$ when $\tau = 0$. Under regularity conditions, we expect Ξ_{AE} to become more negative as τ increases, while Ξ_{RD} will become less positive, which yields a well-behaved solution for τ.

1.2 Random earnings

Environment. We now consider an environment in which individual earnings are partly due to a random process involving luck, in addition to ability and effort. We still assume that individuals exclusively consume during a single period in which they have to a make consumption-labor decision. We assume that individuals have expected utility preferences of the form

$$
\int u^i \left(c^i \left(\varepsilon \right), n^i \left(\varepsilon \right) \right) dF \left(\varepsilon | i \right),
$$

where ε denotes a shock to individual abilities, so the wage is now a function of ε, as in $w^i\left(\varepsilon\right)$. We assume that the distribution of the shock ε can be different for different individuals, according to $dF\left(\varepsilon\right)|i$. In this case, an individual's budget constraint takes the form

$$
c^i \left(\varepsilon \right) = \left(1 - \tau \right) w^i \left(\varepsilon \right) n^i \left(\varepsilon \right) + g.
$$

We assume that there is a distribution of ex-ante types given by $\nu\left(i\right)$, where $\int d\nu\left(i\right) = 1$.

Individual optimality. Ex-post, the consumption-hours decision of a given individual is identical to the problem without risk. Hence, we can define an indirect utility function for individual i in terms of τ and g as

$$
V_i \left(\tau, g \right) = \max \int u_i \left(c^i \left(\varepsilon \right), n^i \left(\varepsilon \right) \right) dF \left(\varepsilon | i \right) \quad \text{s.t.} \quad c^i \left(\varepsilon \right) = \left(1 - \tau \right) w^i \left(\varepsilon \right) n^i \left(\varepsilon \right) + g.
$$

However, we can write the total change in indirect utility for individual i as

$$\frac{dV_i}{d\tau} = \int \frac{\partial u_i\left(c^i(\varepsilon), n^i(\varepsilon)\right)}{\partial c^i} \underbrace{\left(-w^i(\varepsilon) n^i(\varepsilon) + \frac{dg}{d\tau}\right)}_{=\frac{du_{i|c}(\varepsilon)}{d\tau}} \underbrace{dF(\varepsilon|i)}_{f(\varepsilon|i)d\varepsilon}.$$

Optimal linear income tax. The government chooses τ and g to maximize a particular social welfare function $\mathcal{W}(\cdot)$ subject to a revenue constraint and to the constraints that represent individual behavior. Formally,

$$W(\tau) = \mathcal{W}\left(\{V_i(\tau, g(\tau))\}_{i\in I}\right),$$

where the mapping $g(\tau)$ is defined by the government's budget constraint, which takes the form

$$E = \tau \iint w^i(\varepsilon) n^i\left((1-\tau) w^i(\varepsilon), g\right) dF(\varepsilon|i)\, d\nu(i) - g,$$

where the function $n^i\left((1-\tau) w^i, g\right)$ denotes individual i's Marshallian labor supply. We can write $dF(\varepsilon|i) = f(\varepsilon|i)\, d\varepsilon$.

A welfare assessment for a welfarist planner takes the form

$$\begin{aligned}\frac{dW(\tau)}{d\tau} &= \int \frac{\partial \mathcal{W}}{\partial V_i} \frac{dV_i(\tau, g(\tau))}{d\tau} di \\ &= \int \frac{\partial \mathcal{W}}{\partial V_i} \int \frac{\partial u_i\left(c^i(\varepsilon), n^i(\varepsilon)\right)}{\partial c^i} \left(-w^i(\varepsilon) n^i(\varepsilon) + \frac{dg}{d\tau}\right) f(\varepsilon|i)\, d\varepsilon di.\end{aligned}$$

In this environment, a DS-planner can compute individual and stochastic weights, since there is a single date. The normalized individual and stochastic weights take the form

$$\begin{aligned}\tilde{\omega}^i &= \frac{\frac{\partial \mathcal{W}}{\partial V_i} \int \frac{\partial u_i\left(c^i(\varepsilon), n^i(\varepsilon)\right)}{\partial c^i} f(\varepsilon|i)\, d\varepsilon}{\int \frac{\partial \mathcal{W}}{\partial V_i} \int \frac{\partial u_i(c^i(\varepsilon), n^i(\varepsilon))}{\partial c^i} f(\varepsilon|i)\, d\varepsilon di} \\ \tilde{\omega}^i(\varepsilon) &= \frac{\frac{\partial u_i\left(c^i(\varepsilon), n^i(\varepsilon)\right)}{\partial c^i} f(\varepsilon|i)}{\int \frac{\partial u_i(c^i(\varepsilon), n^i(\varepsilon))}{\partial c^i} f(\varepsilon|i)\, d\varepsilon}.\end{aligned}$$

Given these normalized weights, we can express the aggregate welfare assessment as

$$\frac{\frac{dW(\tau)}{d\tau}}{\iint \frac{\partial \mathcal{W}}{\partial V_i} \frac{\partial u_i(c^i(\varepsilon), n^i(\varepsilon))}{\partial c^i} dF(\varepsilon|i)\, di} = \int \tilde{\omega}^i \int \tilde{\omega}^i(\varepsilon) \left(-w^i(\varepsilon) n^i(\varepsilon) + \frac{dg}{d\tau}\right) d\varepsilon di.$$

As in the deterministic earning model, we can write $\frac{dg}{d\tau} = Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)}$. In this case,

$$\begin{aligned}\frac{du_{i|c}}{d\tau} &= -w^i(\varepsilon) n^i(\varepsilon) + \frac{dg}{d\tau} \\ &= -z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)}.\end{aligned}$$

Hence, using the fact that $z^i(\varepsilon) \equiv w^i(\varepsilon) n^i(\varepsilon)$, we can express a welfare assessment as

$$
\begin{aligned}
\frac{dW^{DS}}{d\tau} &= \int \tilde{\omega}^i \left(\int \tilde{\omega}^i(\varepsilon) \left(-z^i(\varepsilon) + \frac{dg}{d\tau} \right) d\varepsilon \right) di \\
&= \int \tilde{\omega}^i \left(\int \tilde{\omega}^i(\varepsilon) \left(-z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)} \right) d\varepsilon \right) di \\
&= \iint \tilde{\omega}^i(\varepsilon) \left(-z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)} \right) d\varepsilon di \\
&\quad \underbrace{+ \mathbb{C}ov_i \left[\tilde{\omega}^i, \int \tilde{\omega}^i(\varepsilon) \left(-z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)} \right) d\varepsilon \right]}_{=\Xi_{RD} \text{ (Redistribution)}} \\
&= \iint \frac{\tilde{\omega}^i(\varepsilon)}{f(\varepsilon|i)} \left(-z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)} \right) \underbrace{f(\varepsilon|i)\, d\varepsilon di}_{=dF(\varepsilon,i)} + \Xi_{RD} \\
&= \mathbb{E}_{\varepsilon,i} \left[\frac{\tilde{\omega}^i(\varepsilon)}{f(\varepsilon|i)} \left(-z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)} \right) \right] + \Xi_{RD} \\
&= \iint \left(-z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)} \right) f(\varepsilon|i)\, d\varepsilon di \\
&\quad + \mathbb{C}ov_{\varepsilon,i} \left[\tilde{\omega}^i(\varepsilon), -z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)} \right] + \Xi_{RD} \\
&= \underbrace{-\tau \frac{dZ(1-\tau)}{d(1-\tau)}}_{=\Xi_{AE} \text{ (Agg. Efficiency)}} + \underbrace{\mathbb{C}ov_{\varepsilon,i} \left[\frac{\tilde{\omega}^i(\varepsilon)}{f(\varepsilon|i)}, -z^i(\varepsilon) \right]}_{=\Xi_{RS} \text{ (Risk-Sharing)}} + \underbrace{\mathbb{C}ov_i \left[\tilde{\omega}^i, -\int \tilde{\omega}^i(\varepsilon) z^i(\varepsilon)\, dF(\varepsilon|i) \right]}_{=\Xi_{RD} \text{ (Redistribution)}}. \qquad (3)
\end{aligned}
$$

Note that $\iint \tilde{\omega}^i(\varepsilon)\, d\varepsilon di = 1$, since $\int \tilde{\omega}^i(\varepsilon)\, d\varepsilon = 1$. Note that $\mathbb{E}_{\varepsilon,i}\left[\frac{\tilde{\omega}^i(\varepsilon)}{f(\varepsilon|i)}\right] = \iint \frac{\tilde{\omega}^i(\varepsilon)}{f(\varepsilon|i)} f(\varepsilon|i)\, d\varepsilon di = 1$. Note also that $\iint f(\varepsilon|i)\, d\varepsilon di = 1$ and that it is critical that there is no aggregate-risk in this economy. Typically, we have that $\Xi_{AE} < 0$, while $\Xi_{RD} > 0$. The sign of Ξ_{RS} is ambiguous, and depends on the joint distribution of shocks and ex-ante heterogeneity.

Note that the redistribution component can be further decomposed, following the subdecomposition in Section 6.1 of Dávila and Schaab (2022), as

$$
\begin{aligned}
\Xi_{RD} &= \mathbb{C}ov_i \left[\tilde{\omega}^i, \int \tilde{\omega}^i(\varepsilon) \left(-z^i(\varepsilon) + Z - \tau \frac{dZ(1-\tau)}{d(1-\tau)} \right) d\varepsilon \right] \\
&= \mathbb{C}ov_i \left[\tilde{\omega}^i, -\int \tilde{\omega}^i(\varepsilon) z^i(\varepsilon)\, dF(\varepsilon|i) \right] \\
&= \mathbb{C}ov_i \left[\tilde{\omega}^i, -\left(\int z^i(\varepsilon)\, dF(\varepsilon|i) + \mathbb{C}ov_i \left[\tilde{\omega}^i(\varepsilon), z^i(\varepsilon) \right] \right) \right] \\
&= \underbrace{\mathbb{C}ov_i \left[\tilde{\omega}^i, -\int z^i(\varepsilon)\, dF(\varepsilon|i) \right]}_{\text{Expected Redistribution}} + \underbrace{\mathbb{C}ov_i \left[\tilde{\omega}^i, \mathbb{C}ov_i \left[\tilde{\omega}^i(\varepsilon), -z^i(\varepsilon) \right] \right]}_{\text{Redistributive Smoothing}}.
\end{aligned}
$$

Summary of new insights.

- The equity-efficiency tradeoff in the random earnings case is different to the equity-efficiency tradeoff in the deterministic earnings case.
- Welfare assessments and hence the optimal tax in the stochastic earnings case include a new distinct

motive for taxation: risk-sharing.

- Importantly, risk-sharing is part of efficiency and every normalized welfarist planner will perceive the exact same value of Ξ_{RS}.[3] The same cannot be said about the redistribution component, Ξ_{RD}, which critically hinges on the choice of social welfare function for welfarist planners.
- In particular, if one assumes that all individuals are ex-ante identical in this economy, a welfare assessment boils down to

$$\frac{dW^{DS}}{d\tau} = \underbrace{-\tau \frac{dZ\left(1-\tau\right)}{d\left(1-\tau\right)}}_{=\Xi_{AE} \text{ (Agg. Efficiency)}} + \underbrace{\mathbb{C}ov_{\varepsilon,i}\left[\frac{\tilde{\omega}^{i}\left(\varepsilon\right)}{f\left(\varepsilon|i\right)}, -z^{i}\left(\varepsilon\right)\right]}_{=\Xi_{RS} \text{ (Risk-Sharing)}} \tag{4}$$

- Hence, even if Equations (2) and (5) seem similar, they are very different, in the sense that every normalized welfarist planner will agree with the optimal policy prescription that comes out Equation 2, while different welfarist planners will disagree on the level of the optimal tax implied by Equation (2).

- The optimal tax for a NR (No-Redistribution) DS-Planner will always be $\tau^{\star} = 0$ in the deterministic earnings case, but it will typically be different from zero in the random earnings case.
- The optimal tax for an AE (Aggregate Efficiency) DS-planner will always be $\tau^{\star} = 0$, both in the deterministic and the random earnings cases.

[3] Dávila and Schaab (2022) define efficiency as $\Xi_E = \Xi_{RD} + \Xi_{RS} + \Xi_{IS}$.

2 Application #2: Linear Capital Taxation in the Neoclassical Growth Model with Uninsurable Idiosyncratic Shocks (based on Dávila et al. (2012))

In their highly influential contribution, J. Dávila, J. Hong, P. Krusell and J.V. Ríos-Rull (2012) study the welfare properties of the one-sector neoclassical growth model with uninsurable idiosyncratic shocks and precautionary savings. After illustrating the role played by pecuniary externalities in a version of the model with ex-ante identical individuals, the paper then revisits the main results in an economy with initial wealth heterogeneity. The paper motivates the study of the model with initial wealth heterogeneity by arguing that is better connected to the infinite-horizon studied later:

> *" (...) with sufficient dispersion in initial wealth, it would not be possible to find a Pareto improvement by altering aggregate saving. However, the main point of considering initial wealth inequality here is that it provides a useful link to the analysis of the infinite-horizon model studied in the sections to follow."*

The paper then justifies the use of a utilitarian objective as follows:

> *"Again, the utilitarian objective may seem unmotivated in the two-period model, but the idea, elaborated on above, is that this two-period model represents the last two periods of a longer-horizon problem of which, at time zero, all consumers were equal"*[4]

In what follows, we will formally illustrate that the welfare assessments of a utilitarian in this economy involve aggregate efficiency, risk-sharing, and redistribution considerations, but not intertemporal-sharing. For simplicity, we illustrate our results in the context of varying a linear capital income tax. Similar insights will obtain for other policies.

2.1 Environment

We study the same environment as in Section 2 of Dávila et al. (2012), only augmented to allow for additional individual heterogeneity. In particular, we allow for arbitrary ex-ante heterogeneity in preferences, via β_i and $u_i(\cdot)$, as well as in the initial endowment, via m^i.[5]

This economy is populated by a unit measure of individuals. An individual i solves

$$\max_{c_0^i, c_1^i(\varepsilon), a^i} u_i\left(c_0^i\right) + \beta_i \int u_i\left(c_1^i(\varepsilon)\right) dG_i(\varepsilon),$$

[4] The previous elaboration of the argument is

> *"There, initial (as of time 0) wealth is identical across agents, but as a result of uninsurable earnings shocks, wealth levels will diverge over time; in a laissez-faire steady state, there is a nontrivial joint distribution over asset levels and employment status. Thus, in that setting, as in the model studied in the previous section, there is a natural planner objective, namely, ex ante expected utility—which will be equal for all agents, though realized utility, of course, differs across consumers. Since ex ante expected utility amounts to a probability-weighted average, it can be thought of as a utilitarian objective: the planner is "behind the veil of ignorance." This means that an ex post desire for redistribution from the consumption-rich to the consumption-poor reflects the ex ante insurance aim. Now turning back to our two-period model, based on the previous discussion, we can think of it as the last two periods of a long-horizon model, which then means that the appropriate planner objective is the utilitarian one. Thus, whether more or less aggregate saving is called for in the second-to-last period is more readily answered: we only need to sum the effects on welfare across all consumers."*

[5] Note that the index i in this application exclusively captures ex-ante heterogeneity, as in the previous application and Section G.6 of the Online Appendix in Dávila and Schaab (2022).

subject to budget constraints

$$c_0^i = m^i - (1+\tau)\, a^i + T^i$$
$$c_1^i(\varepsilon) = r \cdot a^i + w \cdot e^i(\varepsilon) + \underbrace{\Pi^i}_{=0},\ \forall \varepsilon.$$

Individuals face idiosyncratic stocks, indexed by ε, distributed according to $dG(\cdot)$.[6] We denote initial wealth by m^i, savings by a^i, and the idiosyncratic realization of labor productivity (or hours) by $e^i(\varepsilon)$. The interest rate and the wage are denoted by r and w, respectively. Consumption is denoted by c_0^i and $c_1^i(\varepsilon)$.

We allow for a linear tax on capital, τ, whose proceeds are rebated according to T^i. We typically consider i) targeted rebates or ii) uniform rebates, that is,

$$T^i = \begin{cases} \tau a^i, & \text{targeted rebate} \\ \tau K, & \text{uniform rebate} \end{cases}$$

In either case, note that $\int T^i di = \tau K$, and this implies that

$$\frac{d \int T^i di}{d\tau} = \int \frac{dT^i}{d\tau} di = K + \tau \frac{dK}{d\tau}.$$

In this economy, aggregate capital and labor are given by

$$K = \int a^i di$$
$$L = \iint e^i(\varepsilon)\, dG_i(\varepsilon)\, di,$$

which are scalars under a law of large numbers. Competitive firms produce at date 1 using a Cobb-Douglas technology.[7] Hence, given the absence of aggregate uncertainty, r and w are pinned down by

$$r = F_K(K, L)$$
$$w = F_L(K, L).$$

The optimality condition of individual i is given by an Euler equation of the form[8]

$$(1+\tau)\, u_i'\left(c_0^i\right) = \beta r \int u_i'\left(c_1^i(\varepsilon)\right) dG_i(\varepsilon) \Rightarrow 1 + \tau = \beta r \int \frac{u_i'\left(c_1^i(\varepsilon)\right)}{u_i'\left(c_0^i\right)} dG_i(\varepsilon). \tag{5}$$

[6] In Dávila et al. (2012), they assume that the idiosyncratic shock ε, can simply take two values,

$$e^i(\varepsilon) = \begin{cases} e^1, & \text{with } \pi\left(e^1\right) \\ e^2, & \text{with } \pi\left(e^2\right) \end{cases}.$$

[7] In a competitive equilibrium, profits are given by

$$\Pi = f(K, L) - rK - wL = (F_L - w)\, L + (F_K - r)\, K = 0.$$

Given the CRS assumption, the distribution of profits is irrelevant, since they are zero.

[8] Below, we will use the fact that

$$(1+\tau)\, \tilde{\omega}_0^i = r \tilde{\omega}_1^i \iff (1+\tau)\, \tilde{\omega}_0^i = r \tilde{\omega}_1^i \int \tilde{\omega}_1^i(\varepsilon)\, d\varepsilon \iff (1+\tau) \int \tilde{\omega}_0^i di = r \int \tilde{\omega}_1^i \int \tilde{\omega}_1^i(\varepsilon)\, d\varepsilon di,$$

since $\int \tilde{\omega}_1^i(\varepsilon)\, d\varepsilon = 1$.

2.2 Welfare Assessments

Useful preliminary computations. Note that

$$\frac{dV^i}{d\tau} = u_i'\left(c_0^i\right)\frac{dc_0^i}{d\tau} + \beta\int u_i'\left(c_1^i\left(\varepsilon\right)\right)\frac{dc_1^i\left(\varepsilon\right)}{d\tau}dG\left(\varepsilon\right),$$

where

$$\begin{aligned}\frac{dc_0^i}{d\tau} &= -\left(1+\tau\right)\frac{da^i}{d\tau} - a^i + \frac{dT^i}{d\tau}\\ \frac{dc_1^i\left(\varepsilon\right)}{d\tau} &= r\frac{da^i}{d\tau} + \frac{dr}{d\tau}a^i + \frac{dw}{d\tau}e^i\left(\varepsilon\right),\end{aligned}$$

and

$$\frac{dT^i}{d\tau} = \begin{cases}\tau\frac{da^i}{d\tau} + a^i, & \text{targeted rebate}\\ \tau\frac{dK}{d\tau} + K, & \text{uniform rebate.}\end{cases}$$

Hence

$$\frac{dc_0^i}{d\tau} = \begin{cases}-\frac{da^i}{d\tau}, & \text{targeted rebate}\\ -\frac{da^i}{d\tau} + K - a^i + \tau\left(\frac{dK}{d\tau} - \frac{da^i}{d\tau}\right), & \text{uniform rebate.}\end{cases}$$

and

$$\int\frac{dc_0^i}{d\tau}di = \underbrace{-\int\left(1+\tau\right)\frac{da^i}{d\tau}di}_{=-(1+\tau)\frac{dK}{d\tau}} - \underbrace{\int a^i di}_{=K} + \underbrace{\int\frac{dT^i}{d\tau}di}_{K+\tau\frac{dK}{d\tau}} = -\frac{dK}{d\tau}$$

$$\begin{aligned}\iint\frac{dc_1^i\left(\varepsilon\right)}{d\tau}dF\left(\varepsilon,i\right) &= r\iint\frac{da^i}{d\tau}dF\left(\varepsilon,i\right) + \iint\frac{dr}{d\tau}a^i dF\left(\varepsilon,i\right) + \iint\frac{dw}{d\tau}e^i\left(\varepsilon\right)dF\left(\varepsilon,i\right)\\ &= r\underbrace{\iint\frac{da^i}{d\tau}dF\left(\varepsilon,i\right)}_{=\frac{dK}{d\tau}} + \frac{dr}{d\tau}\underbrace{\iint a^i dF\left(\varepsilon,i\right)}_{=K} + \frac{dw}{d\tau}\underbrace{\iint e^i\left(\varepsilon\right)dF\left(\varepsilon,i\right)}_{=L}\\ &= r\frac{dK}{d\tau}.\end{aligned}$$

The last line follows from the zero-profit condition, since

$$\frac{d\Pi}{d\tau} = \left(F_L - w\right)\frac{dL}{d\tau} + \left(F_K - r\right)\frac{dK}{d\tau} - K\frac{dr}{d\tau} - L\frac{dw}{d\tau} = -\left(K\frac{dr}{d\tau} + L\frac{dw}{d\tau}\right) = 0.$$

The fact that the *distributive pecuniary effects* of a policy change, $K\frac{dr}{d\tau} + L\frac{dw}{d\tau}$, add up to zero is a manifestation of a more general result: see Equation (20) and the associated discussion in Dávila and Korinek (2018).

Finally, Dávila et al. (2012) show that

$$\frac{dr}{d\tau} > 0 \quad \text{and} \quad \frac{dw}{d\tau} < 0.$$

DS-weights definition. Using the definition of DS-planner introduced in Dávila and Schaab (2022), and the individual multiplicative decomposition introduced in Lemma 1 of that paper, we can write the aggregate welfare assessments of a change in the capital tax τ as follows:

$$\frac{dW}{d\tau} = \int \tilde{\omega}^i \left(\tilde{\omega}_0^i \frac{dc_0^i}{d\tau} + \tilde{\omega}_1^i \int \tilde{\omega}_1^i (\varepsilon) \frac{dc_1^i (\varepsilon)}{d\tau} d\varepsilon \right) di.$$

The individual component of DS-weights (for utilitarian planner) is given by[9]

$$\tilde{\omega}^i = \frac{u_i' \left(c_0^i\right) + \beta \int u_i' \left(c_1^i (\varepsilon)\right) dF (\varepsilon)}{\int \left(u_i' \left(c_0^i\right) + \beta \int u_i' \left(c_1^i (\varepsilon)\right) dF (\varepsilon)\right) di}.$$

The dynamic component of DS-weights is given by

$$\tilde{\omega}_0^i = \frac{u_i' \left(c_0^i\right)}{u_i' \left(c_0^i\right) + \beta \int u_i' \left(c_1^i (\varepsilon)\right) dF (\varepsilon)}$$
$$\tilde{\omega}_1^i = \frac{\beta \int u_i' \left(c_1^i (\varepsilon)\right) dF (\varepsilon)}{u_i' \left(c_0^i\right) + \beta \int u_i' \left(c_1^i (\varepsilon)\right) dF (\varepsilon)}.$$

For any value of τ, note that Equation (5) implies that $\tilde{\omega}_0^i$ and $\tilde{\omega}_1^i$ are identical across individuals.

Finally, the stochastic component of DS-weights is given by

$$\tilde{\omega}_1^i (\varepsilon) = \frac{u_i' \left(c_1^i (\varepsilon)\right) \frac{dF(\varepsilon)}{d\varepsilon}}{\int u_i' \left(c_1^i (\varepsilon)\right) dF (\varepsilon)}.$$

Aggregate additive decomposition. The aggregate additive decomposition of the welfare assessments associated with changing the capital tax τ takes the form

$$\begin{aligned}
dW &= \int \tilde{\omega}^i \left(\tilde{\omega}_0^i \frac{dc_0^i}{d\tau} + \tilde{\omega}_1^i \int \tilde{\omega}_1^i (\varepsilon) \frac{dc_1^i (\varepsilon)}{d\tau} d\varepsilon \right) di \\
&= \int \left(\tilde{\omega}_0^i \frac{dc_0^i}{d\tau} + \tilde{\omega}_1^i \int \tilde{\omega}_1^i (\varepsilon) \frac{dc_1^i (\varepsilon)}{d\tau} d\varepsilon \right) di + \underbrace{\mathbb{C}ov_i \left[\tilde{\omega}^i, \tilde{\omega}_0^i \frac{dc_0^i}{d\tau} + \tilde{\omega}_1^i \int \tilde{\omega}_1^i (\varepsilon) \frac{dc_1^i (\varepsilon)}{d\tau} d\varepsilon \right]}_{=\Xi_{RD}} \\
&= \int \tilde{\omega}_0^i di \int \frac{dc_0^i}{d\tau} di + \int \tilde{\omega}_1^i di \iint \tilde{\omega}_1^i (\varepsilon) \frac{dc_1^i (\varepsilon)}{d\tau} d\varepsilon di + \underbrace{\mathbb{C}ov_i \left[\tilde{\omega}_0^i, \frac{dc_0^i}{d\tau} \right] + \mathbb{C}ov_i \left[\tilde{\omega}_1^i, \int \tilde{\omega}_1^i (\varepsilon) \frac{dc_1^i (\varepsilon)}{d\tau} d\varepsilon di \right]}_{=\Xi_{IS}} + \Xi_{RD} \\
&= \int \tilde{\omega}_0^i di \int \frac{dc_0^i}{d\tau} di + \int \tilde{\omega}_1^i di \left(\iint \frac{\tilde{\omega}_1^i (\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}} \frac{dc_1^i (\varepsilon)}{d\tau} \underbrace{dF (\varepsilon) di}_{dF(\varepsilon, i)} \right) + \Xi_{IS} + \Xi_{RD} \\
&= \int \tilde{\omega}_0^i di \int \frac{dc_0^i}{d\tau} di + \int \tilde{\omega}_1^i di \left(\iint \frac{\tilde{\omega}_1^i (\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}} dF (\varepsilon, i) \iint \frac{dc_1^i (\varepsilon)}{d\tau} dF (\varepsilon, i) + \mathbb{C}ov_{\varepsilon, i} \left[\frac{\tilde{\omega}_1^i (\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}}, \frac{dc_1^i (\varepsilon)}{d\tau} \right] \right) + \Xi_{IS} + \Xi_{RD} \\
&= \underbrace{\int \tilde{\omega}_0^i di \int \frac{dc_0^i}{d\tau} di + \int \tilde{\omega}_1^i di \iint \frac{\tilde{\omega}_1^i (\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}} dF (\varepsilon, i) \iint \frac{dc_1^i (\varepsilon)}{d\tau} dF (\varepsilon, i)}_{=\Xi_{AE}} + \underbrace{\int \tilde{\omega}_1^i di \mathbb{C}ov_{\varepsilon, i} \left[\frac{\tilde{\omega}_1^i (\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}}, \frac{dc_1^i (\varepsilon)}{d\tau} \right]}_{=\Xi_{RS}} + \Xi_{IS} + \Xi_{RD}.
\end{aligned}$$

[9] It is straightforward to consider an arbitrary welfarist planner.

Summing up, we have

$$\Xi_{AE}=\int\tilde{\omega}_0^i di\int\frac{dc_0^i}{d\tau}di+\int\tilde{\omega}_1^i di\iint\frac{\tilde{\omega}_1^i(\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}}dF(\varepsilon,i)\iint\frac{dc_1^i(\varepsilon)}{d\tau}dF(\varepsilon,i)$$

$$\Xi_{RS}=\int\tilde{\omega}_1^i di\mathbb{C}ov_{\varepsilon,i}\left[\frac{\tilde{\omega}_1^i(\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}},\frac{dc_1^i(\varepsilon)}{d\tau}\right]$$

$$\Xi_{IS}=\mathbb{C}ov_i\left[\tilde{\omega}_0^i,\frac{dc_0^i}{d\tau}\right]+\mathbb{C}ov_i\left[\tilde{\omega}_1^i,\int\tilde{\omega}_1^i(\varepsilon)\frac{dc_1^i(\varepsilon)}{d\tau}d\varepsilon di\right]$$

$$\Xi_{RD}=\mathbb{C}ov_i\left[\tilde{\omega}^i,\tilde{\omega}_0^i\frac{dc_0^i}{d\tau}+\tilde{\omega}_1^i\int\tilde{\omega}_1^i(\varepsilon)\frac{dc_1^i(\varepsilon)}{d\tau}d\varepsilon\right].$$

Exploring the different components. We now provide insights into each of the components of the aggregate additive decomposition.

Redistribution. First, we study the redistribution term, Ξ_{RD}. Note that

$$\begin{aligned}\tilde{\omega}_0^i\frac{dc_0^i}{d\tau}+\tilde{\omega}_1^i\int\tilde{\omega}_1^i(\varepsilon)\frac{dc_1^i(\varepsilon)}{d\tau}d\varepsilon&=\tilde{\omega}_0^i\left(-(1+\tau)\frac{da^i}{d\tau}-a^i+\frac{dT^i}{d\tau}\right)+\tilde{\omega}_1^i\int\tilde{\omega}_1^i(\varepsilon)\left(r\frac{da^i}{d\tau}+\frac{dr}{d\tau}a^i+\frac{dw}{d\tau}e^i(\varepsilon)\right)d\varepsilon\\&=\tilde{\omega}_0^i\left(-a^i+\frac{dT^i}{d\tau}\right)+\tilde{\omega}_1^i\int\tilde{\omega}_1^i(\varepsilon)\left(\frac{dr}{d\tau}a^i+\frac{dw}{d\tau}e^i(\varepsilon)\right)d\varepsilon\\&\quad+\underbrace{\left[-(1+\tau)\tilde{\omega}_0^i+r\tilde{\omega}_1^i\int\tilde{\omega}_1^i(\varepsilon)d\varepsilon\right]}_{=0}\frac{da^i}{d\tau}=0\\&=\tilde{\omega}_0^i\left(-a^i+\frac{dT^i}{d\tau}\right)+\tilde{\omega}_1^i\int\tilde{\omega}_1^i(\varepsilon)\left(\frac{dr}{d\tau}a^i+\frac{dw}{d\tau}e^i(\varepsilon)\right)d\varepsilon,\end{aligned}$$

where

$$\tilde{\omega}_0^i\left(-a^i+\frac{dT^i}{d\tau}\right)=\begin{cases}\tilde{\omega}_0^i\tau\frac{da^i}{d\tau}, & \text{targeted rebate}\\ \tilde{\omega}_0^i\left(\tau\frac{dK}{d\tau}+K-a^i\right), & \text{uniform rebate}\end{cases}$$

Hence, with the targeted rebate, the non-pecuniary component of Ξ^{RD} at $\tau=0$ is zero. With the uniform rebate, individuals with assets below average, $K-a^i>0$, benefit from the increase in the tax, and vice versa. In general, the redistribution term has three components:

1. $\tilde{\omega}_0^i\left(K-a^i\right)$: related to the presence of a non-targeted rebate, favoring individuals with low assets
2. $\tilde{\omega}_0^i\tau\frac{da^i}{dk}$: related to the losses associated with taxation and how the may differentially affect different individuals (this term is 0 when $\tau=0$)
3. $\tilde{\omega}_1^i\int\tilde{\omega}_1^i(\varepsilon)\left(\frac{dr}{d\tau}a^i+\frac{dw}{d\tau}e^i(\varepsilon)\right)d\varepsilon$: these are the distributive pecuniary effects of the policy

Intertemporal-sharing. Second, note that the intertemporal-sharing term is always zero, since $\tilde{\omega}_0^i$ and $\tilde{\omega}_1^i$ are identical for all individuals.

Risk-sharing. Third, we study the risk-sharing term, Ξ_{RS}. In this case, note that

$$\Xi_{RS}=\int\tilde{\omega}_1^i di\mathbb{C}ov_{\varepsilon,i}\left[\frac{\tilde{\omega}_1^i(\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}},r\frac{da^i}{d\tau}+\frac{dr}{d\tau}a^i+\frac{dw}{d\tau}e^i(\varepsilon)\right].$$

Interestingly, the $r\frac{da^i}{d\tau}$ term impact risk-sharing, even though individuals have an envelope condition on a^i.

Note that if agents are ex-ante identical, then $\tilde{\omega}_1^i(\varepsilon)$ is constant across i's, and a^i and $\frac{da^i}{d\tau}$ is the same across individuals, so

$$\Xi_{RS} = \int \tilde{\omega}_1^i di \mathbb{C}ov_{\varepsilon,i}\left[\frac{\tilde{\omega}_1(\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}}, \frac{dw}{d\tau}e^i(\varepsilon)\right].$$

Since $\frac{dw}{d\tau} < 0$, and agents with high $e^i(\varepsilon)$ have low $\tilde{\omega}_1^i(\varepsilon)$, it is straightforward to show that $\Xi_{RS} > 0$ in this case.

Aggregate Efficiency. Finally, the aggregate efficiency term can be written as

$$\Xi_{AE} = \int \tilde{\omega}_0^i di \underbrace{\int dc_0^i di}_{=-\frac{dK}{d\tau}} + \int \tilde{\omega}_1^i di \iint \frac{\tilde{\omega}_1^i(\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}} dF(\varepsilon, i) \underbrace{\iint \frac{dc_1^i(\varepsilon)}{d\tau} dF(\varepsilon, i)}_{=r\frac{dK}{d\tau}}.$$

So the aggregate efficiency term satisfies an aggregate Euler equation of the form

$$\begin{aligned}
\Xi_{AE} &= \left[-\int \tilde{\omega}_0^i di + r\int \tilde{\omega}_1^i di \iint \frac{\tilde{\omega}_1^i(\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}} dF(\varepsilon, i)\right]\frac{dK}{d\tau} \\
&= \left[-\int \tilde{\omega}_0^i di + r\int \tilde{\omega}_1^i di \iint \frac{\tilde{\omega}_1^i(\varepsilon)}{\frac{dF(\varepsilon)}{d\varepsilon}} dF(\varepsilon) di\right]\frac{dK}{d\tau} \\
&= \left[-\int \tilde{\omega}_0^i di + r\int \tilde{\omega}_1^i di \iint \tilde{\omega}_1^i(\varepsilon) d\varepsilon di\right]\frac{dK}{d\tau}.
\end{aligned}$$

If markets were complete, this equation would be zero. But what is the sign of this term if markets are incomplete? In general, we can substitute in the individual Euler equations to find[10]

$$\Xi_{AE} = \int \tilde{\omega}_0^i di \tau \frac{dK}{d\tau}.$$

From here, we can conclude that introducing a positive capital tax always has negative impact on aggregate efficiency, since $\frac{dK}{d\tau} < 0$ and $\tau > 0$. Hence, an AE DS-planner always finds that the optimal

Summary of new insights.

1. It is always the case that the intertemporal-sharing term $\Xi_{IS} = 0$, because all agents can freely trade in capital, which in this economy is a risk-free security due to the absence of aggregate risk.

2. The redistribution term Ξ_{RS} cannot be immediately signed, and depends on:

   (a) How the tax revenues are rebated,

   (b) The potential differential impact of tax distortions among individuals (this effects is 0 when $\tau = 0$),

   (c) The distributive pecuniary effects of the policy.

3. When individuals are ex-ante identical, it is easy to show that the risk-sharing term is positive, so $\Xi_{RS} > 0$.

[10]By adding up the individual Euler equations, we find that

$$\tau \int \tilde{\omega}_0^i di = -\int \tilde{\omega}_0^i di + r\int \tilde{\omega}_1^i \int \tilde{\omega}_1^i(\varepsilon) d\varepsilon di.$$

4. When $\tau = 0$, it is the case that $\Xi_{AE} = 0$. But when $\tau > 0$, we have that Ξ_{AE}. Hence this model yields a simple theory of capital taxation. An increase in τ is beneficial in terms of risk-sharing, but this is costly in terms of aggregate efficiency. When agents are ex-ante identical, this is the only tradeoff. If agents are ex-ante heterogeneous, there may be other considerations that impact the desirability of a policy change.

5. An AE (Aggregate Efficiency) DS-Planner — see Section 5 of Dávila and Schaab (2022) — always finds that the optimal capital tax is 0.

6. A NR (No-Redistribution) DS-Planner — which exclusively maximizes efficiency and perceives that $\Xi_{RD} = 0$ — will find that a positive optimal capital tax is optimal.

3 Application #3: Optimal Deposit Insurance with Heterogeneous Depositors (based on Dávila and Goldstein (2021))

In this application, we include a proof of Proposition 9 in Dávila and Goldstein (2021), which in turn provides an alternative characterization — based on DS-weights — of Equation (26) in Proposition 2 of that paper. That paper characterizes the optimal level of deposit insurance coverage in an economy in which banks have depositors that are ex-ante heterogeneous. We use the exact some notation as in that paper.

This application illustrates how to work with DS-weights in the context of instantaneous social welfare functions — studied in Section 6.4 of Dávila and Schaab (2022) — as well as how to use DS-weights in environments with multiple equilibria.

3.1 Derivation

We start from an instantaneous social welfare function, so $ISWF = \int V(j, \delta, R_1)\, dH(j)$, where

$$\begin{aligned} V(\tau, \delta, R_1) = &\int_{\underline{s}}^{\hat{s}(R_1)} \zeta(j,s)\, U\left(C^F(\tau, s)\right) dF(s) \\ &+ \int_{\hat{s}(R_1)}^{s^*(\delta, R_1)} \left(\pi \zeta(j,s)\, U\left(C^F(\tau,s)\right) + (1-\pi)\, \zeta(j,s)\, U\left(C^N(\tau,s)\right)\right) dF(s) \\ &+ \int_{s^*(\delta,R_1)}^{\overline{s}} \zeta(j,s)\, U\left(C^N(\tau,s)\right) dF(s), \end{aligned}$$

where $\zeta(j,s)$ denotes instantaneous Pareto weights. Given this ISWF, we can express $\frac{dV(j,\delta,R_1)}{d\delta} = \frac{d\mathbb{E}_s[\zeta(j,s)U(C_t(j,s))]}{d\delta}$ as

$$\begin{aligned} \frac{dV(j,\delta,R_1)}{d\delta} = &\overbrace{\int_{\underline{s}}^{\hat{s}} \zeta(j,s)\, U'\left(C_t^F(j,s)\right) \frac{\partial C_t^F(j,s)}{\partial \delta} dF(s) + \pi \int_{\hat{s}}^{s^*} \zeta(j,s)\, U'\left(C_t^F(j,s)\right) \frac{\partial C_t^F(j,s)}{\partial \delta} dF(s)}^{= q^F \mathbb{E}_s^F \left[\zeta(j,s) U'\left(C_t^F(j,s)\right) \frac{\partial C_t^F(j,s)}{\partial \delta}\right]} \\ &+ \left[\zeta^-(j,s^*)\, U\left(C_t^F(j,s^*)\right) - \zeta^+(j,s^*)\, U\left(C_t^N(j,s^*)\right)\right] \pi f(s^*) \frac{\partial s^*}{\partial \delta}, \end{aligned}$$

Now, transforming instantaneous Pareto weights (defined over utilities) into dynamic stochastic weights (defined over consumption) we can express $\frac{dV(j,\delta,R_1)}{d\delta}$ as

$$\frac{d\tilde{V}(j,\delta,R_1)}{d\delta} = q^F \mathbb{E}_s^F \left[\omega_t(j,s) \frac{\partial C_t^F(j,s)}{\partial \delta}\right] + \left[\omega_t^F(j,s^*)\, C_t^F(j,s^*) - \omega_t^N(j,s^*)\, C_t^N(j,s^*)\right] \frac{\partial q^F}{\partial \delta},$$

where we define $\frac{\partial q^F}{\partial \delta} = \pi f(s^*) \frac{\partial s^*}{\partial \delta}$, $\omega_t(j,s) = \zeta(j,s)\, U'\left(C_t^F(j,s)\right)$, $\omega_t^F(j,s^*) = \zeta^-(j,s^*) \frac{U\left(C_t^F(j,s^*)\right)}{C_t^F(j,s^*)}$, and $\omega_t^N(j,s^*) = \zeta^+(j,s^*) \frac{U\left(C_t^N(j,s^*)\right)}{C_t^N(j,s^*)}$. We can decompose the dynamic stochastic weights into an individual component and a stochastic component, as follows:

$$\omega_t^N(j,s^*) = \tilde{\omega}(j)\, \tilde{\omega}_t^N(j,s^*), \quad \omega_t^F(j,s^*) = \tilde{\omega}(j)\, \tilde{\omega}_t^F(j,s^*), \quad \omega_t(j,s) = \tilde{\omega}(j)\, \tilde{\omega}_t(j,s).$$

Hence, under the assumption that $\int dH(j) = 1$, which may require a normalization, we can now express a welfare assessment $\frac{dW}{d\delta}$ as follows:

$$\frac{dW}{d\delta} = \int \tilde{\omega}(j) \frac{d\tilde{V}(j,\delta,R_1)}{d\delta} dH(j) = \mathbb{E}_j \left[\tilde{\omega}(j) \frac{d\tilde{V}(j,\delta,R_1)}{d\delta} \right] = \int \frac{d\tilde{V}(j,\delta,R_1)}{d\delta} dH(j) + \Xi_{RD},$$

where we use the fact that $\int dH(j) = 1$ and where

$$\Xi_{RD} = \mathbb{C}ov_j \left[\tilde{\omega}(j), \frac{d\tilde{V}(j,\delta,R_1)}{d\delta} \right]. \tag{6}$$

Note that we can write $\int \frac{d\tilde{V}(j,\delta,R_1)}{d\delta} dH(j) = \mathbb{E}_j \left[\frac{d\tilde{V}(j,\delta,R_1)}{d\delta} \right]$ as

$$\begin{aligned}
\int \frac{d\tilde{V}(j,\delta,R_1)}{d\delta} dH(j) &= -\frac{\partial q^F}{\partial \delta} \left(\mathbb{E}_j \left[\tilde{\omega}_t^N (j,s^*) C_t^N (j,s^*) \right] - \mathbb{E}_j \left[\tilde{\omega}_t^F (j,s^*) C_t^F (j,s^*) \right] \right) \\
&\quad + q^F \mathbb{E}_j \left[\mathbb{E}_s^F \left[\tilde{\omega}_t (j,s) \frac{\partial C_t^F (j,s)}{\partial \delta} \right] \right] \\
&= \Xi_{AE} + \Xi_{RS},
\end{aligned}$$

where

$$\begin{aligned}
\Xi_{AE} &= -\frac{\partial q^F}{\partial \delta} \left(\mathbb{E}_j \left[\tilde{\omega}_t^N (j,s^*) \right] \mathbb{E}_j \left[C_t^N (j,s^*) \right] - \mathbb{E}_j \left[\tilde{\omega}_t^F (j,s^*) \right] \mathbb{E}_j \left[C_t^F (j,s^*) \right] \right) \\
&\quad + q^F \mathbb{E}_s^F \left[\mathbb{E}_j \left[\tilde{\omega}_t (j,s) \right] \mathbb{E}_j \left[\frac{\partial C_t^F (j,s)}{\partial \delta} \right] \right] \qquad (7) \\
\Xi_{RS} &= -\frac{\partial q^F}{\partial \delta} \left(\mathbb{C}ov_j \left[\tilde{\omega}_t^N (j,s^*), C_t^N (j,s^*) \right] - \mathbb{C}ov \left[\tilde{\omega}_t^F (j,s^*), C_t^F (j,s^*) \right] \right) \\
&\quad + q^F \mathbb{E}_s^F \left[\mathbb{C}ov_j \left[\tilde{\omega}_t (j,s), \frac{\partial C_t^F (j,s)}{\partial \delta} \right] \right]. \qquad (8)
\end{aligned}$$

When $\tilde{\omega}(j) = 1$, $\Xi_{RD} = 0$. And when $\tilde{\omega}_t^N (j,s^*) = \tilde{\omega}_t^F (j,s^*)$ and $\tilde{\omega}_t (j,s) = 1$, $\Xi_{RS} = 0$ and Ξ_{AE} is exactly given by

$$\frac{dW}{d\delta} = -\frac{\partial q^F}{\partial \delta} \int \left(C^N (j,s^*) - C^F (j,s^*) \right) dH(j) + q^F \mathbb{E}_s^F \left[\int \frac{\partial C^F (j,s)}{\partial \delta} dH(j) \right], \tag{9}$$

which is the exact counterpart of Equation (26) in Proposition 2.

4 Application #4: Inequality and Welfare (based on Antràs, De Gortari and Itskhoki (2017))

Antràs, De Gortari and Itskhoki (2017) study the welfare implications of trade opening in an environment in which trade increases both aggregate income and income inequality. In this note, we show how their analysis can be mapped to the framework recently developed in Dávila and Schaab (2022). This note focuses on the general argument in Section 2 of Antràs, De Gortari and Itskhoki (2017).

Environment Let's consider a single-period single-good economy populated by unit measure of individuals. Individuals are indexed by φ, with a distribution H_φ, which captures different ability/earnings levels. In general, φ can index any form of individual heterogeneity.[11] Real disposable income (or equivalently consumption) for individual φ is given by

$$r_\varphi^d = \left[1 - \tau\left(r_\varphi\right)\right] r_\varphi + T_\varphi,$$

where aggregate income R must satisfy (under a balanced budget) that

$$R = \int r_\varphi^d dH_\varphi$$

We assume that individuals have preferences of the form

$$u\left(c_\varphi^d\right) = u\left(r_\varphi^d\right),$$

where consumption equals disposable income.

Welfare Assessments It is useful to start with the welfarist approach, in which social welfare W is derived from a Social Welfare Function (SWF). A general arbitrary SWF, denoted by $\mathcal{W}(\cdot)$, takes individual utilities as inputs and defines a social welfare objective W, given by

$$W = \mathcal{W}\left(\underbrace{\left\{u\left(r_\varphi^d\right)\right\}_\varphi}_{=V_\varphi}\right),$$

where we use V_φ to denote the indirect utility of individual φ. As in Dávila and Schaab (2022), we will use a marginal approach to assess welfare changes.[12] Let's index any change in primitives by a one-dimensional parameter θ.[13]

We can thus express a marginal welfare assessment as

$$\frac{dW}{d\theta} = \int \frac{\partial \mathcal{W}(\cdot)}{\partial V_\varphi} u'\left(r_\varphi^d\right) \frac{dr_\varphi^d}{d\theta} dH_\varphi.$$

[11] The index φ in Antràs, De Gortari and Itskhoki (2017) maps to i in the baseline model of Dávila and Schaab (2022).

[12] As explained in Dávila and Schaab (2022), marginal assessments can translated in global assessments by integration. There are different valid approaches to do so.

[13] As explained in Dávila and Schaab (2022), this approach can be used to consider arbitrary multidimensional changes of primitives.

Normalizing $\frac{dW}{d\theta}$ by the social marginal valuation of a transfer to all individuals is simply a choice of units for W that allows us to express welfare assessments as follows:

$$\underbrace{\frac{\frac{dW}{d\theta}}{\int \frac{\partial \mathcal{W}(\cdot)}{\partial V_\varphi} u'\left(r_\varphi^d\right) dH_\varphi}}_{=\frac{dW^{\mathcal{W}}}{d\theta}} = \int \underbrace{\frac{\frac{\partial \mathcal{W}(\cdot)}{\partial V_\varphi} u'\left(r_\varphi^d\right)}{\int \frac{\partial \mathcal{W}(\cdot)}{\partial V_\varphi} u'\left(r_\varphi^d\right) dH_\varphi}}_{=\tilde{\omega}_\varphi} \frac{dr_\varphi^d}{d\theta} dH_\varphi = \int \tilde{\omega}_\varphi \frac{dr_\varphi^d}{d\theta} dH_\varphi = \mathbb{E}_\varphi \left[\tilde{\omega}_\varphi \frac{dr_\varphi^d}{d\theta}\right],$$

where $\mathbb{E}_\varphi[\cdot]$ denotes a cross-sectional average and where

$$\tilde{\omega}_\varphi \equiv \frac{\frac{\partial \mathcal{W}(\cdot)}{\partial V_\varphi} u'\left(r_\varphi^d\right)}{\int \frac{\partial \mathcal{W}(\cdot)}{\partial V_\varphi} u'\left(r_\varphi^d\right) dH_\varphi}$$

exactly corresponds to the individual component of DS-weights for the normalized welfarist planner introduced in Proposition 5 of Dávila and Schaab (2022).[14] Note that, by virtue of the normalization

$$\mathbb{E}_\varphi\left[\tilde{\omega}_\varphi\right] = \int \tilde{\omega}_\varphi dH_\varphi = 1.$$

Therefore, using the aggregate additive decomposition of welfare assessments in Proposition 1 of Dávila and Schaab (2022), we can decompose the welfare effects of any change in primitives for any welfarist planners in this economy into i) Aggregate Efficiency and ii) Redistribution as follows

$$\begin{aligned}
\frac{dW^{\mathcal{W}}}{d\theta} &= \mathbb{E}_\varphi\left[\tilde{\omega}_\varphi \frac{dr_\varphi^d}{d\theta}\right] \\
&= \underbrace{\mathbb{E}_\varphi\left[\tilde{\omega}_\varphi\right]}_{=1} \mathbb{E}_\varphi\left[\frac{dr_\varphi^d}{d\theta}\right] + \mathbb{C}ov_i\left[\tilde{\omega}_\varphi, \frac{dr_\varphi^d}{d\theta}\right] \\
&= \underbrace{\mathbb{E}_\varphi\left[\frac{dr_\varphi^d}{d\theta}\right]}_{=\Xi_{AE}\text{ (Agg. Efficiency)}} + \underbrace{\mathbb{C}ov_i\left[\tilde{\omega}_\varphi, \frac{dr_\varphi^d}{d\theta}\right]}_{=\Xi_{RD}\text{ (Redistribution)}}.
\end{aligned} \tag{10}$$

Since this is a static economy, intertemporal sharing and risk-sharing are necessarily 0 — see Corollary 4 of Proposition 2 in Dávila and Schaab (2022).

If instead of using a welfarist approach, one is willing to use an approach based on generalized welfare weights, the primitive object for welfare assessments is not the SWF, but instead the weights $\tilde{\omega}_\varphi$ in the expression

$$\int \tilde{\omega}_\varphi \frac{dr_\varphi^d}{d\theta} dH_\varphi.$$

That is, by using generalized weights, a planner simply postulates the weights $\tilde{\omega}_\varphi$ instead of the SWF $\mathcal{W}(\cdot)$.

In this economy, because it is static, the individual component of the individual multiplicative decomposition of DS-weights in Dávila and Schaab (2022), denoted by here $\tilde{\omega}_\varphi$, exactly corresponds to the notion of generalized welfare weights introduced in Saez and Stantcheva (2016). In other words, in static

[14] For instance, using isoelastic/CRRA preferences and an equal-weighted utilitarian SWF, as in Antràs, De Gortari and Itskhoki (2017), we can explicitly compute these weights as follows: $\tilde{\omega}_\varphi = \frac{\left(r_\varphi^d\right)^{-\rho}}{\int \left(r_\varphi^d\right)^{-\rho} dH_\varphi} = \frac{\left(r_\varphi^d\right)^{-\rho}}{\mathbb{E}_\varphi\left[\left(r_\varphi^d\right)^{-\rho}\right]}$.

environments like the one considered here, the contribution of Dávila and Schaab (2022) is only to introduce the aggregate additive decomposition of welfare assessments in aggregate efficiency and redistribution, but not to introduce the notion of generalized individual weights, which is already in Saez and Stantcheva (2016).[15]

Insights. Here we make several observations related to Equation (10), and in particular we focus on how it connects to the results in Antràs, De Gortari and Itskhoki (2017).

1. [Kaldor-Hicks approach] In this economy, a marginal welfare assessment under the Kaldor-Hicks principle can be formalized by setting $\tilde{\omega}_\varphi = 1$. In this case, the welfare assessment is purely based on aggregate efficiency. In fact, in this case, the welfare change is simply given by the change in aggregate consumption/disposable income so

$$\frac{dW^{\mathcal{W}}}{d\theta} = \mathbb{E}_\varphi \left[\frac{dr_\varphi^d}{d\theta} \right] = \frac{dR}{d\theta},$$

   where $R = \int r_\varphi^d dH_\varphi$. This is exactly the result in Equation (4) in Antràs, De Gortari and Itskhoki (2017).

2. [Welfarist approach] Formally, the difference between the Kaldor-Hicks approach and the (normalized) welfarist approach simply corresponds to the choice of $\tilde{\omega}_\varphi$:

$$\begin{aligned} \tilde{\omega}_\varphi &= 1 && \text{(Kaldor-Hicks)} \implies \Xi_{RD} = 0 \\ \tilde{\omega}_\varphi &= \frac{u'\left(r_\varphi^d\right)}{\int u'\left(r_\varphi^d\right) dH_\varphi} && \text{(Normalized utilitarian)}, \end{aligned}$$

   where a normalized utilitarian planner is a particular welfarist planner for whom $\frac{\partial \mathcal{W}(\cdot)}{\partial V_\varphi} = 1$. With isoelastic ex-ante identical utilities, as assumed by Antràs, De Gortari and Itskhoki (2017), it follows immediately that $\Xi_{RD} > 0$. It also follows that Ξ_{RD} is increasing in the curvature coefficient ρ. In particular, when $\rho = 0$, the redistribution component is zero: $\Xi_{RD} = 0$. These insights are the counterpart of the discussion of the *welfarist inequality correction* in Antràs, De Gortari and Itskhoki (2017) — see Equation (9) in that paper.

3. [Costly redistribution approach] Antràs, De Gortari and Itskhoki (2017) also account for the fact that actual redistribution among individuals may be costly, since redistributive social insurance systems rely on distortionary taxation. Equation (10) implicitly accounts for this possibility, since $\mathbb{E}_\varphi \left[\frac{dr_\varphi^d}{d\theta} \right]$ already incorporates any potential deadweight losses associated with taxation. As in Applications #1 and #2 in this user guide, increasing taxes typically reduces aggregate efficiency,[16] and this can also have an impact on the redistribution component. Note that Antràs, De Gortari and Itskhoki (2017) formalize these in Equation (15) of their paper, in which they show that the aggregate real income gains need to be adjusted by potential losses from taxation, all of this in addition of accounting for inequality/redistribution.

[15]The central insight is that by using generalized weights is is possible to capture alternative useful welfare notions that are not welfarist, including equality of opportunity or egalitarianism, among others.

[16]See how the aggregate efficiency component in Equations (4) and (9) above becomes more negative as taxes increase.

4. If a welfarist planner had access to lump-sum taxes/transfers, an optimality condition for such a planner is that $\frac{\partial \mathcal{W}(\cdot)}{\partial V_\varphi} u'\left(r_\varphi^d\right)$ must be equal across all agents, implying that $\tilde{\omega}_\varphi = 1$. This is the sense in which $\tilde{\omega}_\varphi = 1$ has the interpretation of Kaldor-Hicks planner. However, while allowing for lump-sum transfers implies that $\tilde{\omega}_\varphi = 1$, the converse is not true, that is, it is possible to making welfare assessments using $\tilde{\omega}_\varphi = 1$ as individual weights even when no transfers at all are made in the background.[17]

5. A key feature of the approach in Dávila and Schaab (2022) is that it systematically generalizes to richer environments. In particular, Equation (10) remains valid for arbitrary individual preferences and arbitrary Social Welfare Functions, while the intuitive and tractable expressions in Antràs, De Gortari and Itskhoki (2017) rely on using a constant-elasticity framework. Moreover, the approach in Dávila and Schaab (2022) can be easily extended to dynamic stochastic models.

[17] If a planner had access to ex-ante transfers, there would be no cost of redistribution either, which would implicitly be reflected in $\mathbb{E}_\varphi\left[\frac{dr_\varphi^d}{d\theta}\right]$, as we discuss above.

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