Welfare Accounting*

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March 2023

Abstract

This paper provides a systematic decomposition of welfare assessments in economies with heterogeneous agents and disaggregated production technologies that accounts for efficiency and redistributional considerations. First, we show that the aggregate efficiency component of a welfare assessment can be decomposed into i) cross-sectional consumption efficiency, ii) cross-sectional factor supply efficiency, iii) cross-sectional intermediate input efficiency, iv) aggregate intermediate input efficiency, v) cross-sectional factor use efficiency, vi) aggregate factor efficiency, vii) technology growth and viii) factor endowment growth. Each of these components exclusively depends on allocations, marginal rates of substitution, and marginal products. Second, we provide the first characterization of efficiency conditions in general disaggregated production economies, extending the results of Lange (1942). These conditions critically depend on network-adjusted social net valuations, which we characterize in terms of a new intermediate inverse matrix. Third, we introduce a new Welfare Hulten's theorem that characterizes the aggregate welfare impact of technology shocks. We also show how to decompose redistribution into distributive pecuniary effects and distortionary effects. Finally, we systematically present the minimal applications that feature each of the six allocative efficiency components of our welfare decomposition. We also introduce a general minimal production economy, which is the simplest economy that captures all welfare-relevant phenomena in disaggregated production economies.

JEL Codes: E61, D60

Keywords: welfare assessments, heterogeneous agents, disaggregated production, input-output networks, Hulten's theorem, Leontief inverse, production efficiency

*We would like to thank Costas Arkolakis, Xavier Gabaix, Zhen Huo, Patrick Kehoe, Sam Kortum, Giuseppe Moscarini, Guillermo Noguera, Michael Peters, Alireza Tahbaz-Salehi, Akos Valentinyi, and Venky Venkateswaran for helpful comments and discussions.

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1 Introduction

Understanding the sources of welfare gains or losses is critical to assess the impact of shocks and the desirability of policy interventions. Nonetheless, this proves to be a challenging endeavor, in particular in realistic economies in which individuals are heterogeneous and production takes place using disaggregated technologies. The consumption baskets and labor supply patterns of heterogeneous individuals can vary significantly, and production processes often rely on multiple factors and a complex network of intermediate inputs.

In light of these complexities, this paper introduces a novel decomposition of welfare assessments that applies to general economies with heterogeneous agents and disaggregated production technologies. We refer to this approach as *welfare accounting* in contrast to traditional growth accounting, an approach that seeks to identify and quantify the importance of various sources of output growth, not welfare. Welfare accounting is useful to identify and quantify the ultimate origins of welfare changes induced by any change in allocations. It can also be used to characterize efficiency conditions and to determine the welfare impact of changes in technologies — via the Welfare Hulten's theorem in frictionless competitive economies — or factor endowments.

Welfare decomposition. The first and central result of this paper is the decomposition of welfare assessments that we illustrate in Figure 1 — see page 14. We sequentially build towards this decomposition in a sequence of steps. In a first step, we leverage the approach of Dávila and Schaab (2022) to decompose welfare assessments for general welfarist planners into an aggregate efficiency and a redistribution component. A central property of this decomposition is that aggregate efficiency is invariant to the choice of social welfare function. In other words, all welfarist planners agree on the definition of aggregate efficiency. In contrast to that paper, which takes the mapping between allocations and policies or shocks as given, in this paper we exploit resource constraints and production technologies to identify the ultimate origins of welfare gains and losses.

Cross-sectional individual efficiency. In a second step, we decompose aggregate efficiency into *cross-sectional individual efficiency* and *production efficiency*. Cross-sectional individual efficiency, which only exists in economies with heterogeneous individuals, captures welfare gains or losses associated with reallocating consumption and factor supplies across individuals to their better uses.[1] We show how to further decompose cross-sectional individual efficiency into cross-sectional consumption efficiency and cross-sectional factor supply efficiency. Intuitively, if holding fixed aggregate consumption and aggregate factor supply, a planner can reallocate

[1]Cross-sectional individual efficiency is a completely different concept from redistribution, even though both notions are only relevant in economies with individual heterogeneity. In particular, the choice of social welfare function does not impact cross-sectional individual efficiency, but it directly impacts redistribution.

consumption or reassign factor supplies to increase welfare, this would be respectively attributed to cross-sectional consumption efficiency and cross-sectional factor supply efficiency.

Production efficiency. Next, we sequentially define and study three different decompositions of production efficiency. First, we present a decomposition of production efficiency into aggregate consumption efficiency and aggregate factor supply efficiency. This decomposition is useful to define a notion of changes in aggregate output/value added/GDP in our environment and to highlight the differences between these notions and changes in welfare. An important takeaway of this decomposition is that we are able to identify a precise notion of aggregate marginal rate of substitution that is the correct object to value changes in final consumption for welfare purposes in economies with individual heterogeneity.

Second, we present a decomposition of production efficiency in levels. This decomposition is useful to understand how production efficiency can be traced back to changes in the level of intermediates inputs, factors, and technologies, allowing us to connect our results to the literature on growth accounting. This decomposition suggests that understanding the network of intermediates is critical to understand the sources of welfare gains and losses, which we do next.

Exclusively relying on production functions, we show that it is possible to express changes in levels of intermediate inputs in terms of changes in shares of intermediates, factor uses, and technology changes. This allows us to capture the network of connections among intermediate inputs, which is critical to capture the restrictions implied by the production structure of the economy. In this context, we introduce the *intermediates inverse matrix,* a new inverse matrix in the space of intermediate input links that accounts for connections among intermediate inputs. This matrix depends exclusively on marginal products and allocations. It plays a central role in defining *network-adjusted social net valuations.* This is a new concept that we introduce to capture the welfare impact of adjusting intermediates, factors, or technologies in economies with disaggregated production structures.

Finally, we present a third and final decomposition of production efficiency in *shares* (instead of levels). In this decomposition, production efficiency can be decomposed into i) cross-sectional intermediate input efficiency, ii) aggregate intermediate input efficiency, iii) cross-sectional factor use efficiency, iv) aggregate factor efficiency, v) technology growth, and vi) factor endowment growth. The decomposition in shares presents the ultimate sources of welfare gains and losses, and is essential to characterize efficiency conditions in Section 6 and the Welfare Hulten's theorem in Section 8.

The first four terms of this decomposition represent different forms of *allocative efficiency*, which capture welfare gains that emerge from changing the allocations of goods or factors. Cross-sectional intermediate input efficiency captures whether there is an improvement in the use of a given intermediate good across different uses, holding fixed the amount of a good not devoted to final consumption. That is, perturbations that increase the share of factors dedicated to uses that have high network-adjusted social net valuations will increase cross-sectional intermediate input efficiency, and vice versa. Aggregate intermediate input efficiency captures instead whether changes in the amount of a good devoted to production versus final consumption are desirable. That is, perturbations that change the share of a good dedicated to production and not final consumption will increase aggregate efficiency as long as the average network-adjusted social net valuation for that good is positive, and vice versa. The same logic applies to factors. Cross-sectional factor use efficiency is positive when a perturbation shifts the proportion of a given factor that goes to goods with higher network-adjusted social net valuations, and vice versa. Aggregate factor efficiency is positive when a perturbation shifts the aggregate amount used of a given factor with a positive average network-adjusted social net valuation, and vice versa.

The last two terms capture the direct welfare impact of changes in technology and factor endowments. Changes in technology growth capture a different form of efficiency, which we refer to as *technical efficiency.* In this case, welfare gains emerge from productivity gains that would increase output even when input allocations are unchanged. Factor endowment growth can also be interpreted as an efficiency gain, which we refer to *endowment use efficiency.* Under the interpretation that factors are in infinite supply, an increase in the endowment of a factor can be thought of shifting the maximum amount of such factor that is suitable for production.

Efficiency conditions. We leverage our decomposition of production efficiency in shares to characterize conditions for efficiency. We formally show that these results can be interpreted as the generalization of Lange (1942) to economies with disaggregated production. In particular, we show how our results nest Lange's when there are no intermediate inputs or when there are no pure intermediate goods.

A central message of this paper is that properly accounting for non-negativity constraints in feasible allocations is critical to properly characterize efficiency and to appropriately understand the origins of welfare gains and losses. This is particularly important when production is disaggregated — since it is well known that production networks are highly sparse, see e.g. Bernard and Zi (2022) — and when individuals are heterogeneous in the sense that they may have disjoint consumption bundles.

Decentralized economies. While the aggregate efficiency decomposition introduced in this paper can be derived without specifying individual budget constraints, imposing individual optimality, or relying on notions of profit maximization or cost minimization, it is useful to understand the properties of our welfare decomposition in decentralized economies. We establish properties of the decomposition that emerge in competitive economies. In particular, we show that i) frictionless competitive economies are efficient, ii) dispersion on wedges on individual consumption and labor supply distort cross-sectional individual efficiency, and iii) that wedges of any form will distort production efficiency. We also show that the intermediates inverse matrix is tightly connected to Leontief inverses in competitive economies. By studying market economies we are able to show how prices can be helpful in recovering the objects we have identified are needed for welfare assessments: marginal rates of substitutions and marginal products.

Welfare Hulten's theorem. Hulten's theorem (Hulten, 1978) is a result that characterizes the change in aggregate output induced by changes in technology. We present a new version of Hulten's theorem formulated in terms of welfare: we hence refer to this result as *Welfare Hulten's theorem*. Welfare Hulten's theorem states that, in frictionless competitive economies, the aggregate efficiency component of welfare assessments associated with a proportional Hicks-neutral technology change is exclusively given by the technology growth component of production efficiency, which can be normalized to correspond to a Domar weight.

Several conclusions follow from our results. First, when there is a single individual, redistribution is zero and aggregate efficiency equals welfare. In that case, the Domar weight exactly captures the welfare change, not only the aggregate efficiency component. Second, we show why Welfare Hulten's theorem applies to economies with elastic supplies, while the standard Output Hulten's theorem fails in that case. Properly accounting for the welfare cost of supplying elastic factors is critical to explain this difference. This result highlights that Hulten's theorem is fundamentally a result about welfare, not about output. Third, we show that both the Welfare Hulten's theorem and the standard Hulten's theorem apply to frictionless competitive economies, instead of efficient economies, as Hulten's theorem is typically stated. The Welfare Hulten's theorem collapses to the standard Hulten's theorem in economies with a single individual and fixed factor supplies.

Redistribution and applications. While most of the paper is focused on understanding the aggregate efficiency component of a welfare assessment, we also explain the determinants of the redistribution component. In this case, we provide two different characterizations of the change in individual well-being for a given individual, which is the key input into the redistribution component. Our second characterization shows that it is possible to express such changes in terms of distributive pecuniary effects (which add up to zero when there is no technology growth) and distortionary effects.

Finally, we conclude by systematically studying the minimal applications that feature each of the components of the decomposition of aggregate efficiency. We study i) a minimal vertical economy; ii) a minimal general production economy; iii) a Robinson Crusoe economy; iv) a minimal horizontal economy; v) a minimal roundabout economy; vi) a minimal diversified intermediate economy; vii) a minimal two-factor supplier economy; and viii) and Edgeworth box economy. We compute our welfare decomposition for each of these economies, and illustrate how the Welfare Hulten's theorem emerges in each case. The minimal general production economy that we introduce is the simplest economy that captures all phenomena associated with production in disaggregated production economies. For that reason, it has the potential to become the workhorse for illustrating phenomena in these environments.

Related literature. When interpreted through the lens of a planning approach, our results are most closely related to the classic work of Lange (1942) — see Section 16.E of Mas-Colell, Whinston and Green (1995) for a modern treatment. Lange provided one of the first proofs of the welfare theorems by first characterizing conditions for efficiency in a planned economy, showing that such conditions are satisfied by a frictionless competitive economy.[2] Lange's approach used an aggregate transformation function — as in Solow (1957) — and modern formulations of his approach (Mas-Colell, Whinston and Green, 1995) assumes that all goods are final. We show that this assumption substantially changes the characterization of efficiency conditions. Our paper provides the first characterization of efficiency conditions in general economies with disaggregated production technologies. As we show in Section 6, our conditions for efficiency collapse to Lange's when we assume away intermediate inputs or when all goods are final. Lange does not provide a decomposition of the source of welfare gains, or losses, which is central to our approach.

Our decomposition of welfare assessments is inspired by the vast literature on growth accounting that follows Solow (1957) and includes Hall and Jones (1999), Basu and Fernald (2002), Petrin and Levinsohn (2012), and Baqaee and Farhi (2020), among many others. At times and to different degrees, part of this body work draws connections between output growth and welfare gains — see, for instance, Basu and Fernald (2002), Basu et al. (2022), or Baqaee and Burstein (2022*a*,*b*). A common challenge for this body of work is to aggregate among heterogeneous individuals. By using the approach introduced in Dávila and Schaab (2022), we are able to make aggregate welfare assessments without having to rely on prices. For instance, it is typical to use Divisia indexes to aggregate, but that form of indexation already uses prices, which our approach does not have to specify. More broadly our paper continues an agenda the seeks to understand the origins of welfare gains and losses in general economies. Note that, at all times, we describe the informational requirements needed to implement our decompositions. While our decompositions are based on marginal rates of substitution and marginal products, which are not directly observable, any measurement approach implicitly connects these objects to observables.

More generally, our results build on the growing literature on multi-sector and production network models, including Gabaix (2011), Jones (2011), Acemoglu et al. (2012), Bigio and Lao (2020), Liu (2019), and Baqaee and Farhi (2018, 2020), among others. See Carvalho and Tahbaz-Salehi (2019) and Baqaee and Rubbo (2022) for recent surveys of this body or work. A central result in this literature is Hulten's theorem Hulten (1978), which we show to be fundamentally a result about welfare by introducing a new Welfare Hulten's theorem that applies to frictionless competitive economies with heterogeneous agents, elastic factor supplies, arbitrary preferences

[2] Lange's proof of the welfare theorems is not the most general — see instead Arrow and Debreu (1954). However, Lange's proof provides many insights into the relation between competition and efficiency. The approach we follow in this paper is subject to the same advantages and disadvantages as Lange's. See Geanakoplos (1989) for a discussion of the different approaches.

and technologies, and arbitrary social welfare functions. A central distinction between these papers and ours is that our main results are derived without making reference to prices, budget constraints, and notions of *behavior*. In particular, our decomposition of welfare assessments is purely based on preferences, technologies, and resource constraints. This contrasts to the decomposition introduced in Baqaee and Farhi (2020), which is based on defining markups and assuming cost minimization by producers. By emphasizing the critical role played by pure intermediate goods for welfare assessments, our results connect to the recent work on global value chains: see Antràs and Chor (2022) for a recent survey.

Finally, our welfare-based approach is related to the work that seeks to refine GDP measurements or replace GDP altogether to generate better measurements of societal welfare. See Nordhaus and Tobin (1973) for an earlier account of these ideas and Fleurbaey (2009) or Basu et al. (2022) for more modern treatments. We hope that our contribution to this debate spurs future measurement efforts.

Outline. Section 2 introduces the baseline environment and introduces several notions that facilitate the presentation of our results. Section 3 first decomposes welfare assessments into aggregate efficiency and redistribution and then decomposes aggregate efficiency into cross-sectional individual efficiency and production efficiency. Section 4 further decomposes cross-sectional individual efficiency into cross-sectional consumption efficiency and cross-sectional factor supply efficiency and establishes their properties. Section 5 sequentially defines and studies three different decompositions of production efficiency. It also characterizes the network of intermediates, as well as the intermediates inverse matrix. Section 6 characterizes efficiency conditions and Section 7 studies market economies with and without wedges. Section 8 introduces a new Welfare Hulten's theorem and Section 9 explores the role redistribution. Section 10 systematically presents the minimal applications that feature each of the six allocative efficiency components of the decomposition of aggregate efficiency. Section 11 concludes. All proofs and derivations are in the Appendix.

2 Environment

We consider a static economy populated by a finite number $I \geq 1$ of individuals, indexed by $i \in \mathcal{I}$, where $\mathcal{I} = \{1, \ldots, I\}$.[3] There are $J \geq 1$ different goods, indexed by $j \in \mathcal{J}$, where $\mathcal{J} = \{1, \ldots, J\}$. There are also $F \geq 0$ different factors, indexed by $f \in \mathcal{F}$, where $\mathcal{F} = \{1, \ldots, F\}$. While goods are produced using goods and factors as inputs, factors are either directly supplied by individuals or appear as an endowment.

[3]In this paper, we exclusively consider static economies with a finite number of individuals, goods and factors. In Dávila and Schaab (2023), we extend the approach of this paper to dynamic stochastic economies. All our results generalize to the case with a continuum of individuals, goods, and factors.

An individual i derives utility from consuming goods and (dis)utility from supplying factors, with a utility function given by

$$V_i = u_i \left(\left\{ c^{ij} \right\}_{j \in \mathcal{J}}, \left\{ n^{if,s} \right\}_{f \in \mathcal{F}} \right), \qquad \text{(Preferences)} \tag{1}$$

where c^{ij} denotes the final consumption of good j by individual i and $n^{if,s}$ denotes the amount of factor f supplied by individual i (the superscript s stands for factor supply).

Goods are produced using technologies that take goods and factors as inputs.[4] The production function for good j is

$$y^j = G^j \left(\left\{ x^{jk} \right\}_{k \in \mathcal{J}}, \left\{ n^{jf,d} \right\}_{f \in \mathcal{F}}; \theta \right), \qquad \text{(Technologies)} \tag{2}$$

where y^j denotes the amount of output of good j, x^{jk} denotes the amount of good k used in the production of good j, and $n^{jf,d}$ denotes the amount of factor f used in the production of good j (the superscript d stands for factor demand). We use the index $k \in \mathcal{J}$ to refer to goods used as intermediates. For clarity, we typically use K to denote the number of intermediate inputs, although $K = J$. The parameter θ allows us to consider perturbations to production technologies, as described below.

We conclude the physical description of the economy by specifying the resource constraints. The resource constraint for good j is

$$y^j = \sum_i c^{ij} + \sum_k x^{kj}, \qquad \text{(Resource Constraints: Goods)} \tag{3}$$

where $c^j = \sum_i c^{ij}$ represents the amount of good j used as final consumption (aggregate consumption), while $x^j = \sum_k x^{kj}$ represents the amount of good j used in the production of other goods. Hence, Equation (3) can also be written as $y^j = c^j + x^j$. The resource constraint for factor f is

$$\sum_i n^{if,s} + \bar{n}^{f,s}(\theta) = \sum_j n^{jf,d}, \qquad \text{(Resource Constraints: Factors)} \tag{4}$$

where $\bar{n}^{f,s}(\theta) \geq 0$ denotes the aggregate endowment of factor f, while $n^{f,s} = \sum_i n^{if,s}$ and $n^{f,d} = \sum_j n^{jf,d}$ respectively represent the aggregate (elastic) factor supply and demand of factor f.[5] Hence, Equation (3) can also be written as $n^{f,s} + \bar{n}^{f,s}(\theta) = n^{f,d}$. We parametrize $\bar{n}^{f,s}(\theta)$ by θ to explicitly consider changes in factor endowments.

[4] A multi-product firm can be interpreted as a subset of production technologies. Alternatively, technologies that produce different goods that individuals treat as perfect substitutes can be interpreted as multiple firms producing the same good.

[5] When needed, we write $\bar{n}^{f,s}(\theta) = \sum_i \bar{n}^{if,s}(\theta)$, where $\bar{n}^{if,s}(\theta)$ denotes the endowment of factor f that belongs to individual i.

Feasible allocation. A *feasible allocation* is defined by $\left\{c^{ij}, n^{if,s}, x^{jk}, n^{jf,d}, y^j\right\}$ such that Equations (2) through (4), as well as non-negativity constraints, that is, $c^{ij} \geq 0$, $n^{if,s} \geq 0$, $x^{jk} \geq 0$, $n^{jf,d} \geq 0$, and $y^j \geq 0$, are satisfied. Binding non-negativity constraints play a central role in our analysis.

Perturbations and assumptions. We assume that preferences and technologies are differentiable and simply impose that all variables are smooth functions of a perturbation parameter $\theta \in [0,1]$, so derivatives such as $\frac{dc^{ij}}{d\theta}$, $\frac{dx^{jk}}{d\theta}$, or $\frac{dn^{if,d}}{d\theta}$ are well-defined. Perturbations, which must be feasible, have a dual interpretation. First, a perturbation may capture changes in technologies or endowments, but also policy changes (e.g., tax changes) or any other change in the primitives of a fully-specified model (e.g., trade costs, markups, etc.) that impact allocations. In this case, the mapping between allocations and the parameter θ typically emerges endogenously and accounts for general equilibrium effects. Second, perturbations may alternatively capture changes in allocations directly chosen by a central planning authority. This second interpretation is useful to characterize efficiency conditions — see Section 6. In either case, our results do not require to specify, for instance, individual budget constraints, firm objectives, or a notion of equilibrium.

Note that our environment i) features heterogeneous individuals, ii) allows for elastic factor supplies, and iii) imposes no assumption on the homotheticity/homogeneity of either utility or production functions. Until Section 7, there is also no reference to prices, which contrasts to existing work on disaggregated production economies.

Finally, to simplify the exposition, we describe our results in the body of the paper assuming that i) consumption is (weakly) desirable but supplying factors is not, i.e., $\frac{\partial u_i}{\partial c^{ij}} \geq 0$ and $\frac{\partial u_i}{\partial n^{if,s}} \leq 0$ and ii) marginal products of intermediates and factors are (weakly) positive, i.e., $\frac{\partial G^j}{\partial x^{jk}} \geq 0$ and $\frac{\partial G^j}{\partial n^{jf,d}} \geq 0$. Our proofs do not impose these restrictions.

Definitions: goods. We introduce several notions that facilitate the presentation of our results. To study production with intermediate inputs, we define the (physical) *jk-intermediate-output share* ξ^{jk} by

$$\xi^{jk} = \frac{x^{jk}}{y^k} = \phi_x^k \chi_x^{jk}, \quad \text{where} \quad \phi_x^k = \frac{x^k}{y^k} \quad \text{and} \quad \chi_x^{jk} = \frac{x^{jk}}{x^k},$$

where we refer to $\phi_x^k = \frac{x^k}{y^k}$ as the (physical) *intermediate share of good k* and to $\chi_x^{jk} = \frac{x^{jk}}{x^k}$ as the (physical) *jk-intermediate-intermediate share*.[6] By construction, $\phi_x^k = \sum_j \xi^{jk} \leq 1$ and $\sum_j \chi_x^{jk} = 1$, while the non-negativity constraints imply that $0 \leq \chi_x^{jk} \leq 1$ and $0 \leq \phi_x^k \leq 1$. Note that these shares are in physical units and do not include any valuation.

In general, a good can be final if used directly by individuals or intermediate if used in the

[6]If $y^k = 0$, we define $\xi^{jk} = \chi_x^{jk} = 0$, and if $x^k = 0$, we define $\chi_x^{jk} = 0$.

production of a different good, but some goods may be both final and intermediate. Hence, based on the intermediate share of good k, ϕ_x^k, we categorize goods in a given allocation as a

i) *pure final good*, when $\phi_x^k = 0$ (so $c^k > 0$ and $x^k = 0$);

ii) *pure intermediate good*, when $\phi_x^k = 1$ (so $c^k = 0$, $x^k > 0$);

iii) *mixed good*, when $\phi_x^k \in (0,1)$ (so $c^k > 0$, $x^k > 0$).

Based on intermediate-intermediate shares χ_x^{jk}, we further distinguish between specialized and diversified intermediates. We categorize intermediate good k in a given allocation as a

i) *specialized intermediate,* when $\chi_x^{jk} = 1$ for some j;

ii) *diversified intermediate,* when $\chi_x^{jk} \in (0,1)$ for some j.

Definitions: factors. To study production with factors, we define the (physical) *jf-factor share* by

$$\chi_n^{jf} = \frac{n^{jf,d}}{n^{f,d}}.$$

By construction, $\sum_j \chi_n^{jf} = 1$. Analogously to the intermediate input case, we distinguish between specialized and diversified factors. We categorize factor f as a

i) *specialized factor,* when $\chi_n^{jf} = 1$ for some j;

ii) *diversified factor* when $\chi_n^{jf} \in (0,1)$ for some j.

3 Welfare Decompositions

In this section, we first describe how a welfarist planner makes welfare assessments in our environment, given a social welfare function. We then provide a decomposition of welfare assessments between aggregate efficiency and redistribution. Finally, we show how aggregate efficiency can be further decomposed into cross-sectional individual efficiency and production efficiency.

3.1 Aggregate Efficiency vs. Redistribution

We use social welfare functions (SWF) to make aggregate welfare assessments, leveraging the welfare decomposition introduced in Dávila and Schaab (2022). Formally, we characterize welfare assessments for a welfarist planner with a SWF given by

$$\mathcal{W}(V_1, \ldots, V_I), \qquad \text{(Social Welfare Function)} \tag{5}$$

where V_i is defined in Equation (1) and where we assume that $\frac{\partial \mathcal{W}}{\partial V_i} > 0$, $\forall i$.[7] The utilitarian SWF, which adds up a weighted sum of individual utilities, is the most used in practice, although there are other commonly used SWF's — see e.g., Kaplow (2011).

Hence, welfare assessments of a welfarist planner with SWF $\mathcal{W}(\cdot)$ can be expressed as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V_i} \frac{dV_i}{d\theta} = \sum_i \alpha^i \lambda^i \frac{\frac{dV_i}{d\theta}}{\lambda^i},$$

where $\alpha_i = \frac{\partial \mathcal{W}}{\partial V_i}$ denotes the marginal social welfare gain of increasing individual's i utility and where we define λ^i as an individual normalizing factor that allows us to express individual welfare assessments into a common unit/numeraire. The choice of λ^i determines the welfare numeraire, that is, the numeraire in which we express welfare assessments. In particular, note that $\dim(\lambda^i) = \frac{\text{utils of individual } i}{\text{units of numeraire}}$, which allows us to express $\frac{\frac{dV_i}{d\theta}}{\lambda^i}$ in a common unit, since $\dim\left(\frac{\frac{dV_i}{d\theta}}{\lambda^i}\right) = \frac{\text{units of numeraire}}{\text{units of } \theta}$ for all i. The only restriction when choosing the welfare numeraire is that λ^i must be strictly positive for all individuals, that is, $\lambda^i > 0$.[8]

Lemma 1 derives the additive aggregate decomposition of welfare assessments for a normalized welfarist planner introduced Dávila and Schaab (2022) in the environment considered here.

Lemma 1. *(Welfare decomposition: aggregate efficiency vs. redistribution) The aggregate welfare assessment of a normalized welfarist planner, $\frac{dW^\lambda}{d\theta}$, can be decomposed into two components: i) aggregate efficiency, Ξ^{AE}, and ii) redistribution, Ξ^{RD}, as follows:*

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{\sum_i \alpha^i \lambda^i}{I}} = \underbrace{\sum_i \frac{\frac{dV_i}{d\theta}}{\lambda^i}}_{=\Xi^{AE}} + \underbrace{\mathbb{C}ov_i^\Sigma \left[\tilde{\omega}^i, \frac{\frac{dV_i}{d\theta}}{\lambda^i}\right]}_{=\Xi^{RD}}, \tag{6}$$

where $\tilde{\omega}^i = \frac{\alpha^i \lambda^i}{\frac{\sum_i \alpha^i \lambda^i}{I}}$, $\alpha^i = \frac{\partial \mathcal{W}}{\partial V_i}$, and where λ^i is *an individual normalizing factor expressed in utils of individual i relative to a common numeraire.*[9]

[7] As in Kaplow (2011) or Saez and Stantcheva (2016), we refer to the use of SWF's — typically traced back to Bergson (1938) and Samuelson (1947) — as the *welfarist* approach. As explained in Kaplow (2011), the critical restriction implied by the welfarist approach is that the social welfare function $\mathcal{W}(\cdot)$ cannot depend on any model outcomes besides individual utility levels. It is possible to extend our results to the case in which $\frac{\partial \mathcal{W}}{\partial V_i} = 0$ for some individuals. The framework in Dávila and Schaab (2022) encompasses welfare objectives more general than the welfarist approach, and it is straightforward to extend our results to those. As explained there, this may be helpful to make global welfare assessments based on equivalent or compensating variations in additive or multiplicative form.

[8] While we derive our results for a general normalizing factor λ^i, the marginal value of wealth, expressed in nominal units (e.g., dollars), is the most natural normalization. In that case, λ^i is measured in $\frac{\text{utils of individual } i}{\text{dollars}}$, and $\frac{\frac{dV_i}{d\theta}}{\lambda^i}$ is measured in $\frac{\text{dollars}}{\text{units of } \theta}$. In particular applications, it may be useful to consider alternative welfare numeraires. For instance, one may choose the marginal utility of consuming a particular good or supplying a particular factor.

[9] Throughout the paper, we denote cross-sectional averages among all individuals by $\mathbb{E}_i[\cdot]$ and cross-sectional

The aggregate efficiency component, Ξ^{AE}, corresponds to the (unweighted) sum of changes in individual utilities expressed in units of the common welfare numeraire. The redistribution component, Ξ^{RD}, corresponds to the cross-sectional covariance-sum of individual weights $\tilde{\omega}^i$ with changes in individual utilities expressed in units of the common numeraire. Individual weights $\tilde{\omega}^i$ capture the social marginal valuation of welfare changes for individual i in units of the common numeraire. They are normalized so that they average to one, so $\mathbb{E}_i\left[\tilde{\omega}^i\right] = \sum_i \tilde{\omega}^i / I = 1$.

Three properties of this decomposition — established in Dávila and Schaab (2022) — are relevant for our subsequent analysis. First, the aggregate efficiency component is *invariant* to the choice of SWF. That is, differences in welfare assessments among welfarist planners are exclusively due to redistribution. Second, given the choice of numeraire via the units of λ^i, the aggregate efficiency component is invariant to increasing transformations of individual's utilities. Third, redistribution is zero when there is a single individual $I = 1$. Moreover, redistribution is also zero when the planner can costlessly implement lump-sum transfers among individuals, which gives Ξ^{AE} a Kaldor-Hicks interpretation. These properties support the view that Ξ^{AE} captures the aggregate welfare impact of any perturbation, while Ξ^{RD} is driven by the differential impact of such perturbation across individuals' overall utility.

3.2 Aggregate Efficiency: Cross-Sectional Individual Efficiency vs. Production Efficiency

In Sections 4 through 8, we study in detail the aggregate efficiency component of the welfare decomposition introduced in Lemma 1. Lemma 2 presents an intermediate result showing that aggregate efficiency can be decomposed into cross-sectional individual efficiency and production efficiency.

Lemma 2. *(Aggregate efficiency decomposition: cross-sectional individual efficiency vs. production efficiency) The aggregate efficiency component of a welfare assessment, Ξ^{AE}, can be decomposed into two components: i) cross-sectional individual efficiency, $\Xi^{AE,I}$, and ii) production efficiency, $\Xi^{AE,P}$, as follows:*

$$\Xi^{AE} = \Xi^{AE,I} + \Xi^{AE,P},$$

where $\Xi^{AE,I}$ is defined in Proposition 1, and where $\Xi^{AE,P}$ can be expressed in three alternative formulations, sequentially introduced in Propositions 3, 4, and 6.

In Section 4 we systematically study and characterize properties of cross-sectional individual efficiency, which in turn can be decomposed into cross-sectional consumption efficiency and

covariances among all individuals by $\mathbb{C}ov_i[\cdot, \cdot\cdot]$. We also define covariance-sums as

$$\mathbb{C}ov_i^{\Sigma}[\cdot, \cdot\cdot] = I \cdot \mathbb{C}ov_i[\cdot, \cdot\cdot].$$

With a continuum of individuals in unit measure, covariance and covariance-sums coincidence, that is, $\mathbb{C}ov_i[\cdot, \cdot\cdot] = \mathbb{C}ov_i^{\Sigma}[\cdot, \cdot\cdot]$.

cross-sectional factor supply efficiency. In Section 5 we do the same with production efficiency, which can be ultimately decomposed into cross-sectional intermediate input efficiency, aggregate intermediate input efficiency, cross-sectional factor use efficiency, aggregate factor efficiency, technology growth, and factor endowment growth.

Figure 1 illustrates the decompositions introduced in Lemmas 1 and 2 and serves as a roadmap for the remainder of this paper until Section 7.

4 Cross-Sectional Individual Efficiency

In this section, we show how to further decompose cross-sectional individual efficiency into consumption and factor supply components. Before studying cross-sectional individual efficiency and production efficiency, we must define individual and aggregate marginal rates of substitution. Our particular definition of aggregate marginal rates of substitution turns out to be the appropriate notion to measure the welfare implications of any perturbation. As we emphasize below, the way in which aggregate marginal rates of substitution are defined critically depends on whether perturbations impact final consumption and factors supplies or not.

Definitions: marginal rates of substitution. We define the *(individual) marginal rate of substitution* for individual i between good j and the numeraire, MRS_c^{ij}, and between factor f and the numeraire, MRS_n^{if}, as

$$MRS_c^{ij} = \frac{\frac{\partial u_i}{\partial c^{ij}}}{\lambda_i} \quad \text{and} \quad MRS_n^{if} = -\frac{\frac{\partial u_i}{\partial n^{if}}}{\lambda_i}. \tag{7}$$

The marginal rate of substitution MRS_c^{ij} measures the value in units of the numeraire for individual i of a marginal increase in consumption of good j. Analogously, MRS_n^{if} measures the cost in units of the numeraire for individual i of a marginal increase in the supply of factor f.

We also define the *aggregate marginal rate of substitution* between good j and the numeraire, $\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]$, and between factor f and the numeraire, $\mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]$, as

$$\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] = \mathbb{E}_{i|\frac{dc^{ij}}{d\theta}\neq 0}\left[MRS_c^{ij}\right] \quad \text{and} \quad \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right] = \mathbb{E}_{i|\frac{dn^{if,s}}{d\theta}\neq 0}\left[MRS_n^{if}\right]. \tag{8}$$

That is, $\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]$ corresponds to the cross-sectional average of MRS_c^{ij} among individuals for whom consumption of good j changes. Hence, it can be interpreted as the average value of increasing consumption of good j. Analogously, $\mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]$ corresponds to the cross-sectional average of MRS_n^{if} among individuals for whom the supply of factor f changes. It can be interpreted as the average cost of increasing the supply of factor f.

Defining aggregate marginal rates of substitution via a conditional expectation (through the $\mathbb{E}_i^{\pm}[\cdot]$ operator) is critical. In particular, it is important to highlight that the aggregate marginal

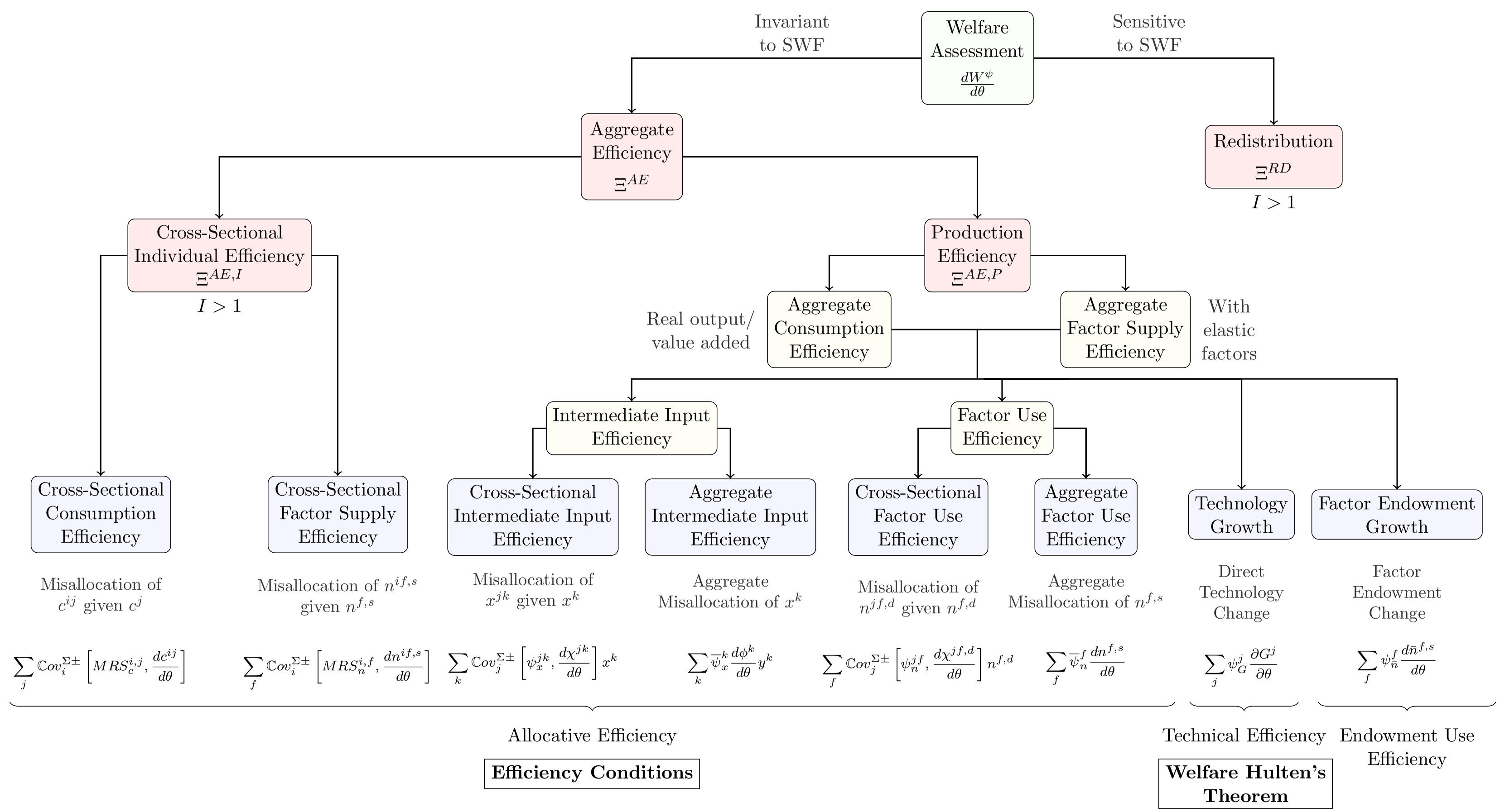


Figure 1: Welfare Decomposition

Note: Figure 1 illustrates the decomposition(s) of welfare assessments introduced in this paper. Lemma 1 decomposes welfare assessments in aggregate efficiency and redistribution. Lemma 2 decomposes aggregate efficiency in cross-sectional individual efficiency and production efficiency. Proposition 1 decomposes cross-sectional individual efficiency in cross-sectional consumption efficiency and cross-sectional factor supply efficiency. Proposition 3 decomposes production efficiency in aggregate consumption efficiency and aggregate factor supply efficiency. Propositions 4 and 6 respectively decompose production efficiency into intermediate input efficiency, factor use efficiency, technology growth, and factor endowment growth in levels (Proposition 4) and shares (Proposition 6). Proposition 6 further decomposes intermediate input efficiency into cross-sectional intermediate input efficiency and aggregate intermediate input efficiency, as well as factor use efficiency into cross-sectional factor use efficiency and aggregate factor efficiency. Propositions 8 and 9 characterize conditions for efficiency. Theorem 1 introduces the Welfare Hulten's theorem.

rate of substitution for a good that is a pure intermediate before and after a perturbation is *zero*. That is, $\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]=0$ since $\frac{dc^{ij}}{d\theta}=0$ for all i, if j is a pure intermediate. Analogously, the aggregate marginal rate of substitution for perturbations that do not impact factor supply is zero. That is, $\mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]=0$ since $\frac{dn^{if,s}}{d\theta}=0$, for all i if f does not impact factor supply. These observations will be central to several of our results.

Cross-sectional individual efficiency. Armed with the definition of individual marginal rates of substitution and Lemma 2, Proposition 1 characterizes the two components of cross-sectional individual efficiency: cross-sectional consumption efficiency and cross-sectional factor supply efficiency.

Proposition 1. *(Cross-sectional individual efficiency) The cross-sectional individual efficiency component of aggregate efficiency, $\Xi^{AE,I}$, can be decomposed into i) cross-sectional consumption efficiency and ii) cross-sectional factor supply efficiency, as follows:*

$$\Xi^{AE,I}=\underbrace{\sum_j \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{ij},\frac{dc^{ij}}{d\theta}\right]}_{\substack{\text{Cross-Sectional}\\ \text{Consumption Efficiency}}}-\underbrace{\sum_f \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_n^{if},\frac{dn^{if,s}}{d\theta}\right]}_{\substack{\text{Cross-Sectional}\\ \text{Factor Supply Efficiency}}}, \tag{9}$$

where $\mathbb{C}ov_i^{\Sigma\pm}[\cdot,\cdot]$ denotes cross-sectional covariance-sums where the second components is non-zero.

Cross-sectional consumption efficiency is positive for good j when a perturbation increases the consumption of good j for those individuals who value good j more — those with a higher MRS_c^{ij}. Analogously, individual factor supply efficiency is positive for factor f when a perturbation increases the factor supply of factor f for those individuals for whom increasing factor supply f is less costly — those with a higher MRS_n^{if}.

Intuitively, if consumption efficiency is positive for good j, this means that it is possible to improve welfare by reallocating consumption among individuals holding fixed the total aggregate consumption of the good, c^j. Analogously, if factor supply efficiency is positive for factor f, this means that it is possible to improve welfare by reallocating factor supply among individuals holding fixed the aggregate (elastic) supply of factor f, $n^{f,s}$. We provide a formal proof of these statements in Section C.1 of the Appendix, where we show that the cross-sectional individual efficiency terms exactly capture the welfare gains from reallocating consumption and factor supplies across individuals.

Properties of cross-sectional individual efficiency. Proposition 2 formalizes three useful properties of cross-sectional individual efficiency.

Proposition 2. *(Properties of cross-sectional individual efficiency)*

a) *(Single individual) In economies with a single individual ($I = 1$), cross-sectional individual efficiency is zero.*

b) *(Equalized MRS_c^{ij}) If marginal rates of substitution for good j are identical across individuals, that is MRS_c^{ij} does not depend on i, for all goods, then cross-sectional consumption efficiency is zero.*

c) *(Equalized MRS_n^{if}) If marginal rates of substitution for factor f are identical across individuals, that is MRS_n^{if} does not depend on i, for all factors, then cross-sectional factor supply efficiency is zero.*

Part a) of Proposition 2 shows that cross-sectional individual efficiency only exists in economies with heterogeneous individuals. Parts b) and c) of Proposition 2 show that cross-sectional variation of marginal rates of substitution across individuals is necessary for cross-sectional individual efficiency to be non-zero. It also follows immediately that only perturbations that impact the consumption or factor supply of individuals differentially may generate welfare gains/losses associated with cross-sectional individual efficiency. A corollary of Proposition 2c) is that in economies in which individuals do not derive (dis)utility from factor supply, cross-sectional individual factor supply efficiency is zero.

It is worth highlighting that cross-sectional individual efficiency is a completely different concept from redistribution, even though both notions are only relevant in economies with individual heterogeneity. In particular, the choice of SWF does not impact cross-sectional individual efficiency, while it directly impacts redistribution.

Informational requirements. From a measurement perspective, knowledge of the distribution of MRS_c^{ij} and MRS_n^{if} as well as changes in individual consumption and factor supply is sufficient to compute cross-sectional individual efficiency. Importantly, conditional these objects, the production structure of the economy does not play an independent role on determining individual cross-sectional efficiency.

5 Production Efficiency

In this Section, we sequentially define and study three different decompositions of production efficiency, introduced in Lemma 2. First, we present a decomposition of production efficiency into aggregate consumption efficiency and aggregate factor supply efficiency. This decomposition is useful to define a notion of changes in aggregate output/value added/GDP in our environment and to highlight the differences between these notions and changes in welfare.

Second, we present a decomposition of production efficiency in *levels*. This decomposition is useful to understand how production efficiency can be traced back to changes in the level of intermediates inputs, factor use, technologies, and factor endowments, allowing us to connect our results to the literature on Growth Accounting. This decomposition suggests that understanding the network of intermediates is critical to understand the sources of welfare gains and losses, Hence, we proceed to relate changes in levels of intermediate inputs to changes in shares of intermediates by characterizing the network of intermediates of the economy. This step is critical to capture the restrictions implied by the disaggregated production structure of the economy.

Finally, we present a third decomposition of production efficiency in *shares* (instead of levels). This production efficiency decomposition in shares is essential to characterize efficiency conditions in Section 6 and the Welfare Hulten's theorem in Section 8.

5.1 Production Efficiency Decomposition: Value Added

Proposition 3 presents the first of the three decompositions of production efficiency that we introduce in this paper. We refer to this decomposition as the value added production efficiency decomposition. This decomposition allows us to highlight the differences between changes in aggregate output/value added/GDP and welfare.

Proposition 3. *(Production efficiency decomposition #1: value added) Production efficiency, $\Xi^{AE,P}$, can be decomposed into two components: i) aggregate consumption efficiency and ii) aggregate factor supply efficiency, as follows:*

$$\Xi^{AE,P} = \underbrace{\sum_{j} \mathbb{E}_{i}^{\pm}\left[MRS_{c}^{ij}\right] \frac{dc^{j}}{d\theta}}_{\substack{\text{Aggregate} \\ \text{Consumption Efficiency}}} - \underbrace{\sum_{f} \mathbb{E}_{i}^{\pm}\left[MRS_{n}^{if}\right] \frac{dn^{f,s}}{d\theta}}_{\substack{\text{Aggregate} \\ \text{Factor Supply Efficiency}}}, \tag{10}$$

where the aggregate marginal rates of substitution $\mathbb{E}_{i}^{\pm}\left[MRS_{c}^{ij}\right]$ and $\mathbb{E}_{i}^{\pm}\left[MRS_{n}^{if}\right]$ are defined in Equation (8).

In this decomposition, aggregate consumption efficiency can be interpreted as the change in aggregate value added or final output, since $\frac{dc^j}{d\theta} = \frac{dy^j}{d\theta} - \frac{dx^j}{d\theta}$, where $\mathbb{E}_{i}^{\pm}\left[MRS_{c}^{ij}\right]$ captures the social valuation of a change in aggregate final consumption of good j. Hence, aggregate consumption efficiency is the natural counterpart of changes in real GDP in this framework, where the aggregate marginal rate of substitution $\mathbb{E}_{i}^{\pm}\left[MRS_{c}^{ij}\right]$ plays the role of prices for the purpose of aggregation. In fact, an important takeaway of this decomposition is that $\mathbb{E}_{i}^{\pm}\left[MRS_{c}^{ij}\right]$ is the correct object to value changes in final consumption for welfare purposes in economies with individual heterogeneity.

In economies with no individual heterogeneity ($I = 1$) and inelastic factor supply, aggregate factor supply efficiency is zero and welfare assessments are exclusively driven by aggregate consumption efficiency (since redistribution and cross-sectional individual efficiency are zero), This fact justifies the use of aggregate consumption efficiency as a welfare measure. However, Equation (10) highlights that accounting for the disutility of supplying factors is necessary to properly account for welfare when factor supply is elastic.

Aggregate factor supply efficiency precisely captures the welfare cost for individuals of supplying factors of production. The need to account for the cost of supplying factors (i.e., leisure) in welfare measures is typically traced back to Nordhaus and Tobin (1973), and is implicitly accounted for in any model that features labor wedges/output gaps, as we show below in our applications. However, it is common in the literature that deals with the aggregate impact of technology changes to exclusively focus on output measures rather than welfare. This discrepancy justifies defining a new Welfare Hulten's theorem, which we introduce in Section 8.

Informational requirements. From a measurement perspective, knowledge of the aggregate marginal rates of substitution for each good and each factor, as well as the change in aggregate final consumption and aggregate factor supplies (or factor demands in for elastic factors, since $\frac{dn^{f,s}}{d\theta} = \frac{dn^{f,d}}{d\theta}$ in that case) is sufficient to compute the value added production efficiency decomposition.

5.2 Production Efficiency Decomposition: Levels

Before introducing the second decomposition of production efficiency, we must define social net valuations for intermediates and factors. Social net valuations are the appropriate concept to measure the value of changes in the use of intermediates and factors. They capture the net social benefit associated with a change in the level of intermediates x^{jk} or factors $n^{jf,d}$.

Definitions: social net valuations We define the *social net valuations* of increasing the level of intermediate inputs x^{jk} and factor use $n^{jf,d}$ by μ_x^{jk} and μ_x^{jf}, respectively. Formally,

$$\mu_x^{jk} = \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{\partial G^j}{\partial x^{jk}} - \mathbb{E}_i^{\pm}\left[MRS_c^{i,k}\right] \qquad \text{Intermediates} \tag{11}$$

$$\mu_n^{jf} = \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{\partial G^j}{\partial n^{jf,d}} - \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right], \qquad \text{Factors} \tag{12}$$

where aggregate marginal rates of substitution are defined in Equation (8), and $\frac{\partial G^j}{\partial x^{jk}}$ and $\frac{\partial G^j}{\partial n^{jf,d}}$ represent marginal products of intermediates and factors, respectively.[10] Intuitively, a marginal increase in x^{jk} generates $\frac{\partial G^j}{\partial x^{jk}}$ units of good j, whose social value is $\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]$, at a social cost of a unit of good k, whose social value is $\mathbb{E}_i^{\pm}\left[MRS_c^{i,k}\right]$. Analogously, a marginal increase in $n^{jf,d}$ generates $\frac{\partial G^j}{\partial n^{jf,d}}$ units of j, whose social value is $\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]$, at a social cost of supplying an extra unit of factor f, whose social value is $\mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]$.

[10]In matrix form, Equations (11) and (12) can be expressed as

$$\boldsymbol{\mu}_x = \boldsymbol{G}_x^T \boldsymbol{MRS}_c - \boldsymbol{1}_x \boldsymbol{MRS}_c$$
$$\boldsymbol{\mu}_n = \boldsymbol{G}_n^T \boldsymbol{MRS}_x - \boldsymbol{1}_n \boldsymbol{MRS}_n,$$

where all matrices are defined in Section A of the Appendix.

Importantly, the social net valuations μ_x^{jk} and μ_n^{jf} can take both positive and negative values. Assuming that marginal products are positive, the sign of the social net valuations depends on whether a particular perturbation changes final consumption for goods j and/or k and whether factor f generates disutility or not. In Equation (13), we describe scenarios in which μ_x^{jk} take different signs. For the purpose of this equation, when we refer to a good as final or pure intermediate it must be so before and after a (generic) perturbation:[11]

$$\mu_x^{jk} = \begin{cases} \gtreqless 0, & \text{if } j \text{ and } k \text{ are final} \\ > 0, & \text{if } j \text{ is final and } k \text{ is pure intermediate} \\ < 0, & \text{if } j \text{ is pure intermediate and } k \text{ is final} \\ = 0, & \text{if } j \text{ and } k \text{ are pure intermediates.} \end{cases} \tag{13}$$

Analogously, in Equation (14), we describe scenarios in which μ_n^{jf} take different signs:

$$\mu_n^{jf} = \begin{cases} \gtreqless 0, & \text{if } j \text{ is final and } k \text{ generates disutility} \\ > 0, & \text{if } j \text{ is final and } k \text{ does not generate disutility} \\ < 0, & \text{if } j \text{ is pure intermediate and } k \text{ generates disutility} \\ = 0, & \text{if } j \text{ is pure intermediate and } k \text{ does not generate disutility.} \end{cases} \tag{14}$$

A central insight from these definitions is that social net valuations are based on i) the aggregate valuation of final goods (more precisely, goods for which final consumption changes in a particular perturbation) and ii) aggregate cost of supplying elastic factors (more precisely, factors for which factor supply changes in a particular perturbation).

Production efficiency decomposition: levels. Exploiting resource constraints and production functions, we present a second decomposition of production efficiency expressed in terms of changes in the level of intermediate inputs, factor use, technology growth, and factor endowment growth. We refer to this decomposition as the production efficiency decomposition in levels. This decomposition is useful to understand how production efficiency can be traced back to changes in the level of intermediates inputs, factor use, technologies, and factor endowments, allowing us to connect our results to the literature on Growth Accounting.

Proposition 4. *(Production efficiency decomposition #2: levels) Production efficiency, $\Xi^{AE,P}$, can be decomposed, when expressed in levels, into i) intermediate input efficiency, ii) factor use*

[11] We include the qualifier generic in the sense that if a good j is a final good before and after the perturbation generically there exists one individual for whom $\frac{dc^{ij}}{d\theta} \neq 0$. Similarly, if factor f generates disutility, generically there exists on individual for whom $\frac{dn^{if,s}}{d\theta} \neq 0$. Equations (13) and (14) are simply illustrations of how social net valuations can take different values, the precise definitions of μ_x^{jk} and μ_n^{jf} are in Equations (11) and (12).

efficiency, iii) technology growth, and iv) factor endowment growth, as follows:

$$\Xi^{AE,P}=\underbrace{\sum_j\sum_k\mu_x^{jk}\frac{dx^{jk}}{d\theta}}_{\substack{\text{Intermediate Input}\\ \text{Efficiency}\\ \text{(Levels)}}}+\underbrace{\sum_j\sum_f\mu_n^{jf}\frac{dn^{jf,d}}{d\theta}}_{\substack{\text{Factor Use}\\ \text{Efficiency}\\ \text{(Levels)}}}+\underbrace{\sum_j\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{\partial G^j}{\partial\theta}}_{\substack{\text{Technology Growth}\\ \text{(Levels)}}}+\underbrace{\sum_f\mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\frac{d\bar{n}^{f,s}(\theta)}{d\theta}}_{\substack{\text{Factor Endowment Growth}\\ \text{(Levels)}}},\tag{15}$$

where social net valuations are defined in Equations (11) and (12). Equivalently, in matrix form

$$\Xi^{AE,P}=\boldsymbol{\mu}_x'\frac{d\boldsymbol{x}}{d\theta}+\boldsymbol{\mu}_n'\frac{d\boldsymbol{n}^d}{d\theta}+\boldsymbol{MRS}_c'\boldsymbol{G}_\theta+\boldsymbol{MRS}_n'\frac{d\bar{\boldsymbol{n}}^{f,s}}{d\theta},\tag{16}$$

where we denote transposes with a ′ and all matrices are defined in Section A of the Appendix.

This decomposition of production efficiency is expressed in terms of changes in inputs of production (intermediate inputs and factors), technology, and factor endowments. Intuitively, production efficiency simply adds up changes in the marginal level of intermediates and factors across all possible jk and jf links, valuing them at their appropriate social net valuation. Moreover, changes in technology, which increase output by $\frac{\partial G^j}{\partial\theta}$, all else equal, are valued at the aggregate MRS of good j, since there is no cost associated with those. Similarly, increases in the supply of factor f are valued the aggregate MRS of such factor, since that the is marginal gain from not having to supply such factor.

Importantly, consistently with the observations made when introducing aggregate marginal rates of substitution and social net valuations, only perturbations that impact final consumption or elastically supplied factors are directly accounted for in this decomposition. For instance, a perturbation that changes the flow of intermediate inputs between two pure intermediate goods (e.g., steel used to produce machinery for a factory) is not directly accounted by Equation (15), since $\mu_x^{jk}=0$ in that case. Similarly, Equation (15) does not directly account for technology changes for pure intermediates, since in that case $\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]=0$. Changes in the endowment of factors that are not elastically supplied by households also do not appear directly in this decomposition, since $\mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]=0$ in that case.[12]

Illustration: minimal vertical economy. At this point, it is helpful to illustrate how the production efficiency decomposition in levels treats pure intermediate goods. We do so in a minimal vertical economy, which we present in more detail on page 41 in Section 10 — see also Figure 2. In this economy, there is a single final good (good 1) produced using a single intermediate good (good 2), which is in turn produced using a single factor in fixed supply.

[12] At first glance, the decomposition in levels introduced in Equation (15) may seem identical to the decomposition in Petrin and Levinsohn (2012). There are two crucial differences. First, our decomposition does not rely on prices of goods or factors. Second, and more importantly, our definitions of aggregate marginal rates of substitution and social net valuations carefully deal with pure intermediates, from which individuals derive no direct value.

In this case, assuming that $\frac{\partial G^1}{\partial \theta} \neq 0$ and $\frac{\partial G^2}{\partial \theta} \neq 0$, the production efficiency decomposition in levels is given by

$$\Xi^{AE,P} = \mu_x^{12} \frac{dx^{12}}{d\theta} + MRS_c^{11} \frac{\partial G^1}{\partial \theta},$$

where $\mu_x^{12} = MRS_c^{11} \frac{\partial G^1}{\partial x^{12}} > 0$. Importantly, note that $\frac{\partial G^2}{\partial \theta}$ does not directly appear in the decomposition, which may seem to suggest that changes in the technology of good 2 do not impact welfare. This is not true. In this particular economy, $\frac{dx^{12}}{d\theta} = \frac{\partial G^2}{\partial \theta}$, which illustrates how changes in the level of intermediates $\frac{dx^{jk}}{d\theta}$ depend on changes in upstream technology, and more generally upstream intermediate inputs and factors too. Hence, the decomposition in levels, while correct, may not always be desirable, because it attributes to intermediates changes that are ultimately caused by changes in technology or factor endowments. This example highlights that it is valuable to further characterize the network of intermediates.

Informational requirements. From a measurement perspective, knowledge of marginal products and aggregate marginal rates of substitution for each good and each factor, which in turn determine social net valuations, as well as the change in the level of intermediates, factor use, technology changes, and factor endowment growth is sufficient to compute the production efficiency decomposition in levels (of intermediates).

5.3 Intermediates Inverse Matrix

Motivated by the example of the minimal vertical economy, in this subsection we characterize the network of changes in the level of intermediates inputs, $\frac{dx^{jk}}{d\theta}$.

Using the definition of the jk-intermediate-output share, $x^{jk} = \xi^{jk} y^k$, we can express the change in the level of intermediate inputs $\frac{dx^{jk}}{d\theta}$ in terms of the change in the intermediate-output share, holding the amount of output of good k fixed, and the change in output of the upstream good k, holding the share ξ^{jk} fixed, as follows:

$$\frac{dx^{jk}}{d\theta} = \frac{d\xi^{jk}}{d\theta} y^k + \xi^{jk} \frac{dy^k}{d\theta}. \tag{17}$$

Using the production function for good k to substitute in Equation (17) for the change in output of the upstream good k, $\frac{dy^k}{d\theta}$, in Proposition 5 we characterize the response of the level of intermediate inputs in terms of changes in intermediate-output shares, factor use, and technology.

Proposition 5. *(Intermediate input network/Intermediate inverse matrix) Changes in the levels of intermediate inputs, $\frac{dx^{jk}}{d\theta}$, can be expressed in terms of changes in intermediate-output shares,*

$\frac{d\xi^{jk}}{d\theta}$, *changes in factor use* $\frac{dn^{kf,d}}{d\theta}$, *and changes in technology,* $\frac{\partial G^k}{\partial\theta}$, *as*

$$\frac{dx^{jk}}{d\theta} = \underbrace{\xi^{jk}\sum_j \frac{\partial G^k}{\partial x^{kj}}\frac{dx^{kj}}{d\theta}}_{Propagation} + \underbrace{\frac{d\xi^{jk}}{d\theta}y^k + \xi^{jk}\left(\sum_f \frac{\partial G^k}{\partial n^{kf,d}}\frac{dn^{kf,d}}{d\theta} + \frac{\partial G^k}{\partial\theta}\right)}_{Impulse}.$$

Equivalently, in matrix form,

$$\frac{d\boldsymbol{x}}{d\theta} = \underbrace{(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1}}_{Propagation}\underbrace{\left(\frac{d\boldsymbol{\xi}}{d\theta}\boldsymbol{y} + \boldsymbol{\xi}\left(\boldsymbol{G}_n\frac{d\boldsymbol{n}}{d\theta} + \boldsymbol{G}_\theta\right)\right)}_{Impulse}, \tag{18}$$

where $\frac{d\boldsymbol{x}}{d\theta}$ *corresponds to the vectorized matrix of* $\frac{dx^{jk}}{d\theta}$ *of dimension* $JK \times 1$, *where the intermediate inverse matrix* $(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1}$ *corresponds to a* $JK \times JK$ *matrix, and the remaining matrices are defined in Section A of the Appendix.*

By expressing changes in the levels of intermediates, $\frac{dx^{jk}}{d\theta}$, in terms of changes in intermediate-output shares, factor use, and technology, Equation (18) accomplishes two objectives. First, it allows us to trace changes in the level of intermediate inputs to more primitive changes, in particular changes in technology — Equation (18) is the generalization of $\frac{dx^{12}}{d\theta} = \frac{\partial G^2}{\partial\theta}$ in the example of the minimal vertical economy just described. Second, as we will show below, when characterizing efficient allocations in this economy it is straightforward to account for non-negativity constraints through $\frac{d\xi^{jk}}{d\theta}$, but not through $\frac{dx^{jk}}{d\theta}$. Intuitively, the planner can easily consider feasible perturbations by considering changes in the intermediate-output shares ξ^{jk}, which are bounded between 0 and 1, but not in x^{jk}, even though $\frac{dx^{jk}}{d\theta}$ and $\frac{d\xi^{jk}}{d\theta}$ are related via Equation (18). After all, Equation (18) is simply a useful way to encode the technological properties of an economy.

The impulse term in Equation (18) can be interpreted as the direct effect of a change in the network of intermediates. First, let's consider a change in technology. Such a change increases output by $\boldsymbol{G}_\theta$, which in turn increases the level of intermediate inputs via $\boldsymbol{\xi}$. Second, a change in the use of factors changes output by $\boldsymbol{G}_n\frac{d\boldsymbol{n}}{d\theta}$, which again increases the level of intermediates via $\boldsymbol{\xi}$. Third, a change in the intermediate-output shares changes directly the levels of intermediates by $\frac{d\boldsymbol{\xi}}{d\theta}\boldsymbol{y}$. All of these changes translate into final changes in the level of intermediate inputs via the inverse matrix $(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1}$, which captures the propagation of an initial impulse in the level of intermediates through the network. Given its novelty and its importance for the characterization of efficiency conditions, we study the matrix $(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1}$, which we refer to as the intermediate inverse matrix, next.

Interpretation of Intermediate Inverse Matrix $(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1}$. The *intermediate inverse matrix* $(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1}$ is an inverse in the space of intermediate input links that accounts for connections among intermediate inputs. A natural interpretation of of this inverse follows from its series representation:

$$(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1} = \boldsymbol{I}_{JK} + \boldsymbol{\xi}\boldsymbol{G}_x + (\boldsymbol{\xi}\boldsymbol{G}_x)^2 + (\boldsymbol{\xi}\boldsymbol{G}_x)^3 + \cdots,$$

where its first element can be interpreted as the first-round effect in the final level of intermediates after a change in the impulse term, the second term as how the first-round impulse changes output, by $\boldsymbol{G}_x$, which in turn generates more intermediates according to $\boldsymbol{\xi}$, and so on so forth. A useful interpretation of the intermediate inverse matrix comes after applying the *Woodbury matrix identity*, a result we present in Lemma 23.

Lemma 3. *(Intermediate inverse: Woodbury matrix identity representation) The intermediate inverse matrix (of dimensions $JK \times JK$) can be expressed as follows:*

$$(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1} = \boldsymbol{I}_{JK} + \boldsymbol{\xi}\left(\boldsymbol{I}_J - \boldsymbol{G}_x\boldsymbol{\xi}\right)^{-1}\boldsymbol{G}_x, \tag{19}$$

where $\boldsymbol{I}_{JK}$ and $\boldsymbol{I}_J$ respectively denote $JK \times JK$ and $J \times J$ identities, $\boldsymbol{\xi}$ corresponds to a $JK \times J$ matrix, and $\boldsymbol{G}_x$ corresponds to a $J \times JK$ matrix, all defined in Section A of the Appendix.

Intuitively, a change in the level of intermediates generates an impulse in term of goods given by $\boldsymbol{G}_x$, which propagates via the input-output network of goods according to

$$(\boldsymbol{I}_J - \boldsymbol{G}_x\boldsymbol{\xi})^{-1} = \boldsymbol{I}_J + \boldsymbol{G}_x\boldsymbol{\xi} + (\boldsymbol{G}_x\boldsymbol{\xi})^2 + (\boldsymbol{G}_x\boldsymbol{\xi})^3 + \cdots,$$

which translates back into intermediates according to $\boldsymbol{\xi}$.[13] From a computational perspective, Equation (19) simplifies computing a $JK \times JK$ inverse into a substantially smaller $J \times J$ inverse. Importantly, the intermediate inverse matrix (of dimensions $JK \times JK$) is *not* a Leontief inverse (of dimensions $J \times J$) either in terms of quantities or values. However, an alternative use of Woodbury's matrix identity implies that:

$$(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1} = \boldsymbol{I}_{JK} + \boldsymbol{\xi}\hat{\boldsymbol{y}}\left(\boldsymbol{I}_J - \hat{\boldsymbol{y}}^{-1}\boldsymbol{G}_x\boldsymbol{\xi}\hat{\boldsymbol{y}}\right)^{-1}\hat{\boldsymbol{y}}^{-1}\boldsymbol{G}_x,$$

where the $J \times J$ matrix $\hat{\boldsymbol{y}}^{-1}\boldsymbol{G}_x\boldsymbol{\xi}\hat{\boldsymbol{y}} = \left\{\frac{\partial \log G^j}{\partial \log x^{jk}}\right\}$ corresponds to the matrix of marginal product elasticities. In Section 8, we show how the matrix $\left(\boldsymbol{I}_J - \hat{\boldsymbol{y}}^{-1}\boldsymbol{G}_x\boldsymbol{\xi}\hat{\boldsymbol{y}}\right)^{-1}$ actually becomes either a Leontief inverse or a cost-based Leontief inverse in particular circumstances.

[13] Intuitively, $(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1}$ maps intermediates into intermediates, which is useful for the network-adjusted social net valuation of intermediates, see Equation (20) below; $(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x)^{-1}\boldsymbol{\xi}$ maps output into intermediates, which is useful for the network-adjusted social net valuation of intermediates, see Equation (21) below; $(\boldsymbol{I}_J - \boldsymbol{G}_x\boldsymbol{\xi})^{-1}$ maps output into output, which connects to Leontief inverses in Section 7; and $(\boldsymbol{I}_J - \boldsymbol{G}_x\boldsymbol{\xi})^{-1}\boldsymbol{G}_x$ maps intermediates into output, as in Equation (19).

5.4 Production Efficiency Decomposition: Shares

Before introducing our third and final decomposition of production efficiency, we must define network-adjusted social net valuations for intermediates and factors. Network-adjusted social net valuations are the appropriate concept to measure the value of changes in intermediate and factor use or technology accounting for its full impact in the level of intermediates via the network of intermediates.

Definitions: network-adjusted social net valuations. We define *network-adjusted social net valuations* of increasing the intermediate-output share, the level of factor use, and technology by $\boldsymbol{\psi}_x'$, $\boldsymbol{\psi}_x'$, and $\boldsymbol{\psi}_G'$, respectively. Formally, in matrix form:

$$\boldsymbol{\psi}_x' = \boldsymbol{\mu}_x' \left(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x\right)^{-1} \qquad \text{Intermediates} \tag{20}$$

$$\boldsymbol{\psi}_n' = \boldsymbol{\mu}_x' \left(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x\right)^{-1} \boldsymbol{\xi}\boldsymbol{G}_n + \boldsymbol{\mu}_n' \qquad \text{Factors} \tag{21}$$

$$\boldsymbol{\psi}_G' = \boldsymbol{\mu}_x' \left(\boldsymbol{I}_{JK} - \boldsymbol{\xi}\boldsymbol{G}_x\right)^{-1} \boldsymbol{\xi} + \boldsymbol{MRS}_c', \qquad \text{Technology} \tag{22}$$

where $\boldsymbol{\psi}_x'$ is a $JK \times 1$ vector, $\boldsymbol{\psi}_n'$ is a $JF \times 1$ vector, and $\boldsymbol{\psi}_G'$ is a $J \times 1$ vector, all defined in Appendix A. We define the elements of each of these vector by ψ_x^{jk}, ψ_n^{jf} and ψ_G^{j} respectively. Network-adjusted social net valuations capture the total welfare impact of the reallocation of intermediate shares, changes in the level of factors, or changes in technology.

The key distinction between social net valuations, denoted by $\boldsymbol{\mu}$ and introduced in Equations (11) and (12) above, and network-adjusted social net valuations, denoted by $\boldsymbol{\psi}$, is that the former only accounts for the direct impact of a change in intermediates use, factor use, or technology, while the latter accounts for the fact that all of these changes will propagate via the network of intermediates (see Lemma 4). In fact, note that when there are no intermediate inputs ($\boldsymbol{\xi} = 0$), $\boldsymbol{\psi}_n' = \boldsymbol{\mu}_n'$ and $\boldsymbol{\psi}_G' = \boldsymbol{MRS}_c'$, a fact that we will use in Proposition 10.

Production efficiency decomposition: shares Using the characterization of the network of intermediates introduced in Proposition 5, we present a third and final decomposition of production efficiency expressed in shares. We refer to this decomposition as the production efficiency decomposition in shares.

Since this decomposition underlies our characterization of efficient allocations in Section 7, and the Welfare Hulten's theorem in section 8, it is valuable to use intermediate-intermediate shares, χ_x^{jk}, and intermediates shares, ϕ_x^k, instead of intermediate-output share ξ_x^{jk}. Formally, we use the fact that

$$\frac{d\xi^{jk}}{d\theta} = \frac{d\chi_x^{jk}}{d\theta}\phi_x^k + \chi_x^{jk}\frac{d\phi_x^k}{d\theta} \quad \text{and} \quad \frac{dn^{jf,d}}{d\theta} = \frac{d\chi_n^{jf}}{d\theta}n^{f,d} + \chi_n^{jf}\frac{dn^{f,d}}{d\theta}, \tag{23}$$

where we use the definition of intermediate-output shares, $\xi^{jk} = \chi_x^{jk}\phi_x^k$ and factor shares,

$n^{jf,d} = \chi_n^{jf} n^{f,d}$. Intuitively, Equation (23) expresses the change in the intermediate-output share in terms of the change in the intermediate-intermediate share, holding the intermediate share of good k fixed, and the change in the intermediate share of good k, holding the intermediate-intermediate share fixed. Equation (23) also expresses the change in the level of factor jf in terms of the change in the factor share, holding aggregate factor use fixed, and the change in the level of aggregate factor f, holding the factor share fixed.

Proposition 6. *(Production efficiency decomposition #3: shares) Production efficiency, $\Xi^{AE,P}$, can be decomposed, when expressed in shares, into i) cross-sectional intermediate input efficiency, ii) aggregate intermediate input efficiency, iii) cross-sectional factor use efficiency, iv) aggregate factor efficiency, v) technology growth, and vi) factor endowment growth, as follows:*

$$\Xi^{AE,P} = \underbrace{\overbrace{\sum_k \mathbb{C}ov_j^{\Sigma}\left[\psi_x^{jk}, \frac{d\chi_x^{jk}}{d\theta}\right] x^k}^{\substack{\text{Cross-Sectional} \\ \text{Intermediate Input Efficiency}}} + \overbrace{\sum_k \overline{\psi}_x^k \frac{d\phi_x^k}{d\theta} y^k}^{\substack{\text{Aggregate} \\ \text{Intermediate Input} \\ \text{Efficiency}}}}_{\substack{\text{Intermediate Input} \\ \text{Efficiency (shares)}}} + \underbrace{\overbrace{\sum_f \mathbb{C}ov_j^{\Sigma}\left[\psi_n^{jf}, \frac{d\chi_n^{jf}}{d\theta}\right] n^{f,d}}^{\substack{\text{Cross-Sectional} \\ \text{Factor Use Efficiency}}} + \overbrace{\sum_f \overline{\psi}_n^f \frac{dn^{f,s}}{d\theta}}^{\substack{\text{Aggregate} \\ \text{Factor Use} \\ \text{Efficiency}}}}_{\substack{\text{Factor Use} \\ \text{Efficiency (shares)}}}$$

$$+ \underbrace{\sum_j \psi_G^j \frac{\partial G^j}{\partial \theta}}_{\substack{\text{Technology Growth} \\ \text{(Shares)}}} + \underbrace{\sum_f \psi_{\bar{n}}^f \frac{d\bar{n}^{f,s}}{d\theta}}_{\substack{\text{Factor Endowment} \\ \text{Growth (Shares)}}}, \tag{24}$$

where ψ_x^{jk}, ψ_n^{jf}, and ψ_G^j are network-adjusted social net valuations, defined in Equations (20) through (22), $\overline{\psi}_x^k$ is a weighted average of ψ_x^{jk} across uses of k given by $\overline{\psi}_x^k = \sum_j \chi_x^{jk} \psi_x^{jk}$, $\overline{\psi}_n^f$ is a weighted average of ψ_n^{jf} across uses of f given by $\overline{\psi}_n^f = \sum_j \chi_n^{jf,d} \psi_n^{jf}$, and $\psi_{\bar{n}}^f$ defines a network-adjusted social net valuation for factor growth, given by $\psi_{\bar{n}}^f = \overline{\psi}_n^f + \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]$.

Equation (24) shows that it is possible to trace welfare changes due to production efficiency to six different components: cross-sectional and aggregate intermediate input efficiency, cross-sectional and aggregate factor efficiency, technology growth, and factor endowment growth. Importantly, the coefficients on $\frac{\partial G^j}{\partial \theta}$ and $\frac{d\bar{n}^{f,s}}{d\theta}$ are different in the decompositions in levels and shares. The production efficiency decomposition in shares is most helpful to characterize whether an allocation is efficient or not, as we explain next and in Section 6.

Let's first consider intermediate input efficiency. Cross-sectional intermediate input efficiency captures whether there is an improvement in the use of given intermediate good across different uses, holding fixed the amount of good k not devoted to final consumption, x^k. That is, perturbations that increase $\frac{d\chi_x^{jk}}{d\theta}$ towards uses that have a high network-adjusted social net valuations will increase cross-sectional intermediate input efficiency, and vice versa. Aggregate intermediate input efficiency captures instead whether changes in the amount of good k devoted to production versus final consumption are desirable. That is, perturbations that change the intermediate share, $\frac{d\phi_x^k}{d\theta}$, will increase aggregate efficiency as long as $\overline{\psi}_x^k$, which is the average network-adjusted social net valuation for good k is positive, and vice versa.

The same logic applies to factors. Cross-sectional factor use efficiency is positive when a perturbation shifts the proportion of a given factor used in the production of goods with higher network-adjusted social net valuations, ψ_n^{jf}, and vice versa. Aggregate factor efficiency is positive when a perturbation increases the supply of factors with a positive average network-adjusted social net valuation, $\overline{\psi}_n^f$, and vice versa. Importantly, aggregate factor efficiency is measured by change in elastic factor supplies, so factors with inelastic factors by construction feature zero aggregate factor efficiency.

Informational requirements. From a measurement perspective, knowledge of marginal products and aggregate marginal rates of substitution for each good and each factor, as well as the matrix of intermediates, which depends on marginal products and intermediate-output shares which in turn determine network-adjusted social net valuations, as well as the change in intermediates-output shares, factor use, and technology changes is sufficient to compute the production efficiency decomposition in shares (of intermediates).

Properties of production efficiency decomposition. Given their practical importance, we highlight several implications of Proposition 6 in four corollaries. These corollaries are helpful to quickly analyze particular economies, as we do in Section 10. In each of the corollaries, by considering specific economies, we implicitly restrict the values that marginal rates of substitution or marginal products can take.

Proposition 7. *(Properties of production efficiency decomposition in shares)*

a) *(Single Good Economies) In economies with a single good ($J = 1$), cross-sectional and aggregate intermediate input efficiency, and cross-sectional factor use efficiency are zero.*[14]

b) *(No Intermediate/Single Intermediate Input Economies) In economies with no intermediate goods ($\xi^{jk} = 0$), cross-sectional and aggregate intermediate input efficiency are zero.*

c) *(Fixed Factor Economies) In economies in which all factors are fixed ($\frac{dn^{f,s}}{d\theta} = 0$), aggregate factor efficiency is zero.*

d) *(Specialized Intermediate/Factor Economies) In economies in which intermediate goods are specialized, cross-sectional intermediate input efficiency is zero. In economies in which factors are specialized, cross-sectional factor use efficiency is zero.*

These corollaries highlight that the decomposition of production efficiency presented in Proposition 6 satisfies desirable properties.

[14] Note that the case in which there are many goods, so $J > 1$, but all individuals treat them as perfect substitutes is not identical to the single good case.

We would like to conclude our study of aggregate efficiency with two remarks on terminology.

Remark. (Allocative vs. Technical vs. Endowment Use Efficiency) As illustrated by Figure 1, we say that welfare gains associated with the two components of cross-sectional individual efficiency and the first four components of the production efficiency decomposition in shares correspond to *allocative efficiency.* We use this terminology because such welfare gains emerge from changing the allocations of goods or factors. Changes in technology growth capture a different form of efficiency, which we refer to as *technical efficiency.* In this case, welfare gains emerge from productivity gains that would increase output even when input allocations are unchanged. Finally, factor endowment growth can also be interpreted as an efficiency gain, which we refer to *endowment use efficiency.* Under the interpretation that factors are in infinite supply, an increase in the endowment of a factor can be thought of shifting the maximum amount of such factor that is suitable for production. Given these definitions, cross-sectional individual efficiency exclusively captures allocative efficiency, while production efficiency captures allocative efficiency, technical efficiency, an endowment use efficiency.

Remark. (Efficiency vs. Misallocation) Throughout the paper, we systematically use the term efficiency to describe each of the components of the decomposition of aggregate efficiency, with the exception of technology growth and factor endowment growth. That is, we say that a particular perturbation increases aggregate efficiency by increasing, let's say, cross-sectional individual consumption efficiency or aggregate factor efficiency. Alternatively, we could have said that such a perturbation increases aggregate efficiency by reducing cross-sectional individual consumption *misallocation* or aggregate factor *misallocation.*

6 Efficiency Conditions

While aggregate efficiency decomposition can be useful by itself to attribute the sources of welfare gains or losses from a given perturbation, the decomposition can also be used is to transparently characterize the conditions that a efficient allocation must satisfy. To our knowledge, this paper provides the first general characterization of efficiency conditions for disaggregated production economies. At all times, we use a conventional definition of (Pareto) efficiency: we say that an allocation is efficient when there is no perturbation that can make every individual in this economy (weakly) better off.

Cross-sectional individual efficiency. For an allocation to be efficient, no feasible perturbation can improve either cross-sectional individual consumption efficiency or cross-sectional individual factor supply efficiency. For this to be the case, it must be that MRS_c^{ij} are identical across all individuals for whom $c^{ij} > 0$, with MRS_c^{ij} being potentially lower for individuals for whom $c^{ij} = 0$. It must also be that MRS_n^{if} are identical across all individuals for whom $n^{if,s} > 0$, and that MRS_n^{if} are potentially lower for individuals for whom $n^{if,s} = 0$. Proposition

8 formalizes this logic.

Proposition 8. *(Efficiency conditions: cross-sectional individual efficiency) The following conditions are necessary for*

a) cross-sectional consumption efficiency:

$$MRS_c^{ij} = \overline{MRS}_c^j,\ \forall i \text{ s.t. } c^{ij} > 0 \quad \textit{and} \quad MRS_c^{ij} < \overline{MRS}_c^j,\ \forall i \text{ s.t. } c^{ij} = 0;$$

b) cross-sectional factor supply efficiency:

$$MRS_n^{if} = \overline{MRS}_n^f,\ \forall i \text{ s.t. } n^{if,s} > 0 \quad \textit{and} \quad MRS_n^{if} < \overline{MRS}_n^f,\ \forall i \text{ s.t. } n^{if,s} = 0.$$

Any departures from these conditions imply that there are feasible perturbations that increase cross-sectional individual efficiency.

Production efficiency. Moreover, for an allocation to be efficient, no feasible perturbation can improve either intermediate input efficiency or factor use efficiency, as defined in Proposition 6. That is, for an allocation to be efficient, it must be that there are no feasible perturbations that increase cross-sectional intermediate input efficiency or aggregate intermediate input efficiency, that is

$$\sum_k \mathbb{C}ov_j^{\Sigma}\left[\psi_x^{jk}, \frac{d\chi_x^{jk}}{d\theta}\right] x^k = 0 \quad \text{and} \quad \sum_k \overline{\psi}_x^k \frac{d\phi_x^k}{d\theta} y^k = 0; \tag{25}$$

for any perturbation such that $\sum_j \frac{d\chi_x^{jk}}{d\theta} = 0$, where $\frac{d\chi_x^{jk}}{d\theta}$ can only be positive when $\chi_x^{jk} = 0$ or negative when $\chi_x^{jk} = 1$, and $\frac{d\phi_x^k}{d\theta}$ can only be positive when $\phi_x^k = 0$ and negative when $\phi_x^k = 1$. Analogously, it must be that there are no feasible perturbations that increase cross-sectional factor efficiency or aggregate factor efficiency, that is

$$\sum_f \mathbb{C}ov_j^{\Sigma}\left[\psi_n^{jf}, \frac{d\chi_n^{jf}}{d\theta}\right] n^{f,d} = 0 \quad \text{and} \quad \sum_f \overline{\psi}_n^f \frac{dn^{f,s}}{d\theta} = 0; \tag{26}$$

for any perturbations such that $\sum_j \frac{d\chi_n^{jf}}{d\theta} = 0$, where $\frac{d\chi_n^{jf}}{d\theta}$ can only be positive when $\chi_n^{jf} = 0$ or negative when $\chi_n^{jf} = 1$, and $\frac{dn^{f,s}}{d\theta}$ can only be positive when $n^{f,s} = 0$.

Proposition 8 formalizes the resulting conditions associated with these requirements.

Proposition 9. *(Efficiency conditions: production efficiency) The following conditions are necessary for*

a) cross-sectional intermediate input efficiency:

$$\psi_x^{jk} = \overline{\psi}_x^k,\ \forall j \text{ s.t. } \chi_x^{jk} \in (0,1);\ \psi_x^{jk} \leq \psi_x^{j'k},\ \forall j' \text{ s.t. } \chi_x^{jk} = 0;\ \psi_x^{jk} \geq \psi_x^{j'k},\ \forall j' \text{ s.t. } \chi_x^{jk} = 1;$$

b) aggregate intermediate input efficiency:

$$\overline{\psi}_x^k = 0,\ \forall k\ s.t.\ \phi_x^k \in (0,1)\,;\ \overline{\psi}_x^k \leq 0,\ \forall k\ s.t.\ \phi_x^k = 0;\ \overline{\psi}_x^k \geq 0,\ \forall k,\ s.t.\ \phi_x^k = 0;$$

c) cross-sectional factor use efficiency:

$$\psi_n^{jf} = \overline{\psi}_n^f,\ \forall j\ s.t.\ \chi_n^{jf} \in (0,1)\,;\ \psi_n^{jf} \leq \overline{\psi}_n^f,\ \forall j\ s.t.\ \chi_n^{jf} = 0;\ \psi_n^{jf} \geq \psi_n^{j'f},\ \forall j'\ s.t.\ \chi_n^{jf} = 1;$$

d) aggregate factor efficiency:

$$\overline{\psi}_n^f = 0,\ \forall k\ s.t.\ n^{f,s} > 0;\ \overline{\psi}_n^f \geq 0,\ \forall f\ s.t.\ n^{f,s} = 0.$$

While the formal statement of the relevant efficiency conditions for production is somewhat involved, the underlying economics are simple. First, whenever a given good k is used as an intermediate for multiple other goods, it must be that the network-adjusted net valuation of doing so must be the same across all uses. When good j does not use good k as a an intermediate input, it must the network-adjusted social net valuation of such link is less than all possible uses of k as an intermediate. In contrast, if good k is a specialized intermediate for good j, it must be that the network-adjusted social net valuation of such connection is the highest among all possible intermediate links of k. Second, it must be that the aggregate intermediate share for a given good is the appropriate one. For this to be the case, it must be that the aggregate network-adjusted net social valuation of adjusting x^k, $\overline{\psi}_x^k = \sum_j \chi_x^{jk} \psi_x^{jk}$, is zero, unless the good is a pure final good, in which case $\overline{\psi}_x^k$ must be negative, or a pure intermediate, in which case $\overline{\psi}_x^k$ must be positive.

The exact same logic applies to the case of factors, with minimal differences. The conditions for cross-sectional factor use efficiency are identical to the intermediate input case. When a factor is used to produce multiple goods, it must be that the network-adjusted social net valuation of each of the uses of the factor is the same. If a factor is exclusively used to produce a given good, it must be that that there is not other use of that factor with a higher network-adjusted social net valuation. If a factor is not used to produce a good, it must be that the network-adjusted social net valuation of doing so is lower than any of the goods for which that factor is used. Finally, it must be that the aggregate amount of labor is the appropriate to have the efficient intermediate-final mix and the total amount of output of each good. In this case, $\overline{\psi}_n^f = \sum_j \chi_n^{jf} \psi_n^{jf}$ must be zero when the factor can be elastically supplied, but it can be positive individuals are not elastically supplying a factor. In contrast to the case of intermediates, there is in principle no upper bound to the amount supplied of a factor.

Remark. (Accounting for binding non-negativity constraints) Propositions 8 and 9 highlight that dealing with non-negative constraints in both the individual side and the production side is critical to derive the correct efficiency conditions. Non-negativity constraints become more important the more disaggregated the economy. With heterogeneous individuals, typically most individuals will not consume most goods. On the production side, production networks become increasingly sparse at finer levels of disaggregation, so carefully accounting for pure intermediate goods, and specialized intermediate inputs and factors is critical to study the welfare properties of a disaggregated economy, as we illustrate in our applications in Section 10.

Lange (1942) revisited. One of the contributions of this paper is to generalize the conditions for efficiency in Lange (1942) to environments with disaggregated production structures.[15] In Lange's model — see a modern treatment in Section 16.E of Mas-Colell, Whinston and Green (1995) — there are three sets of efficiency conditions: i) equalization of marginal rates of substitutions across individuals, ii) equalization of marginal rates of transformation across technologies, and iii) equalization of marginal rates of substitution with marginal rates of transformation. Proposition 8 is the counterpart of the first set of conditions, while Proposition 9 augments the latter two sets of conditions.

The main difference between the analysis in Lange (1942) and Mas-Colell, Whinston and Green (1995) with respect to ours is the presence of pure intermediate goods.[16] Formally, it is easier to connect our results to those in Lange (1942) when we assume that there are no intermediate goods, that is, $\boldsymbol{\xi} = 0$. It is also helpful to consider the case in which all goods are final In Proposition 10, we describe how our general results specialize to those cases.

Proposition 10. *(Lange (1942) revisited: no intermediates and no pure intermediates)*

a) In economies without intermediate inputs ($\boldsymbol{\xi} = 0$):

i) The decomposition of cross-sectional individual efficiency is identical to the one in Proposition 1. The decomposition of production efficiency is simplifies to

$$\Xi^{AE,P} = \underbrace{\boldsymbol{\mu}_n' \frac{d\boldsymbol{n}^d}{d\theta}}_{\substack{\text{Factor Use} \\ \text{Efficiency}}} + \underbrace{\boldsymbol{MRS}_c' \boldsymbol{G}_\theta}_{\text{Technology Growth}} .$$

ii) Social net valuations and network-adjusted social net valuations (for factors and technology) always coincide:

$$\boldsymbol{\psi}_n = \boldsymbol{\mu}_n \quad \text{and} \quad \boldsymbol{\psi}_G = \boldsymbol{MRS}_c.$$

[15]While Lange (1942) characterizes efficiency conditions, that paper or any subsequent expositions do not present welfare decompositions of the form we introduce in this paper.

[16]In contrast to Lange (1942), who assumes a single transformation function, the exposition in Mas-Colell, Whinston and Green (1995) assumes that there many production technologies. However, Mas-Colell, Whinston and Green (1995) restrict their analysis to the case in which production is "interior". The justify this assumption as follows:

> *"(...) every commodity is both an input and an output of the production process. Because this is unrealistic, we emphasize that no more than expositional ease is involved here. Recall that for expositional ease we are not imposing any boundary constraints on the vectors of inputs/outputs."*

Our characterization of efficiency results in Proposition 9, in which network-adjusted net valuations are central, shows that exclusively considering interior economies yields substantially conclusions.

iii) The conditions that definition cross-sectional factor use efficiency for diversified factors simplify to equalization of marginal products:

$$\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{\partial G^j}{\partial n^{jf,d}}=\mathbb{E}_i^{\pm}\left[MRS_c^{ij'}\right]\frac{\partial G^{j'}}{\partial n^{j'f,d}}.$$

b) In economies in which there no pure intermediates, i.e., all goods are final ($\mathbb{E}_i^{\pm}[\cdot]=\mathbb{E}_i[\cdot]$ and $\mathbb{C}ov^{\Sigma\pm}[\cdot]=\mathbb{C}ov^{\Sigma}[\cdot]$):

i) Social net valuations and network-adjusted social net valuations for intermediates are zero at an efficient allocation:

$$\boldsymbol{\psi}_x=\boldsymbol{\mu}_x=0.$$

ii) Social net valuations and network-adjusted social net valuations for factors and technology coincide at an efficient allocation:

$$\boldsymbol{\psi}_n=\boldsymbol{\mu}_n \quad \textit{and} \quad \boldsymbol{\psi}_G=\boldsymbol{MRS}_c.$$

As expected, the absence of connections between intermediates implies that intermediate input efficiency is zero. More substantively, Proposition 10a) highlights that, when $\boldsymbol{\xi}=0$, social net valuations and their network-adjusted counterparts are identical, so $\boldsymbol{\psi}_n=\boldsymbol{\mu}_n$, which simplifies the computation of factor use efficiency and technology growth. Moreover, Proposition 10 shows that, whenever a factor is diversified, it must be the case that marginal products of such factor (valued according the aggregate marginal rate of substitution) are equalized across uses. This results highlight how cross-sectional factor use efficiency is clearly linked to equalizing marginal products, which is the foundation of the literature on misallocation (Restuccia and Rogerson, 2008; Hsieh and Klenow, 2009).

Proposition 10 shows that there is no rationale for doing network-adjusted valuation at an optimum when all goods are final. This is because at an optimum the optimality condition for intermediate inputs already guarantees that the impact of changes via the network of intermediates has zero social value.

7 Decentralized Economies with and without Wedges

The aggregate efficiency decomposition introduced in Proposition 2 can be derived without specifying individual budget constraints, imposing individual optimality, or relying on notions of profit maximization or cost minimization. Introducing those can be helpful for two reasons. First, we can establish properties of the decomposition that emerge in competitive economies. In particular, we show that i) frictionless competitive economies are efficient, ii) dispersion in wedges on individual consumption and labor supply distort cross-sectional individual efficiency, and iii) wedges of any form will distort production efficiency. We also show that the intermediates inverse matrix is tightly connected to Leontief inverses in competitive economies. Second, by studying market economies we are able to show how prices can be helpful in recovering the objects we have identified are needed for welfare assessments: marginal rates of substitutions and marginal products.

Environment Starting from the environment described in Section 2, we now assume that individuals maximize utility and technologies are operated with the objective of minimizing costs and maximizing profits. In order to allow for distortions, we saturate all choices with wedges.

First, we assume that individual i faces a budget constraint of the form

$$\sum_j p^j \left(1+\tau_c^{ij}\right) c^{ij} = \sum_f w^f \left(1+\tau_n^{if,s}\right) n^{if,s} + \sum_j \zeta^{ij}\pi^j + T^{ij}, \tag{27}$$

where p^j denotes the price of good j, w^f denotes the compensation per unit of factor supplied f, $\zeta^{ij}\pi^j$ denotes the profit associated with the operation of technology j received by individual i, and T^{ij} is a lump-sum rebate/subsidy. Individuals face individual specific consumption wedges $1+\tau_c^{ij}$ and individual specific factor supply wedges $1+\tau_n^{if,s}$.

Second, we assume that technologies are operated to minimize costs, which defines a cost function for technology j given by

$$\mathcal{C}^j\left(y^j; w^f, p^k\right) = \min_{n^{jf,d}, x^{jk}} \sum_f w^f \left(1+\tau_n^{if,d}\right) n^{jf,d} + \sum_k p^k \left(1+\tau_x^{jk}\right) x^{jk}, \tag{28}$$

subject to equation (2), where $1+\tau_n^{if,d}$ denotes technology specific factor wedges and $1+\tau_x^{jk}$ denote technology specific intermediate wedges.

Finally, we assume that the supply of good j can be expressed as the solution to a profit maximization problem, where the profit associated with the operation of technology j is given by

$$\pi^j = \max_{y^j} p^j \left(1+\tau_y^j\right) y^j - \mathcal{C}^j\left(y^j; w^f, p^k\right), \tag{29}$$

where $1+\tau_y^j$ denotes a technology wedge and $\mathcal{C}^j\left(y^j; w^f, p^k\right)$ is defined in equation (28). The definition of competitive equilibrium (with wedges) is standard and we relegate it to the Appendix. Lemma 4 characterizes a competitive equilibrium in this economy.

Competitive equilibrium with and without wedges Lemma 4 characterizes a competitive equilibrium with wedges.

Lemma 4. *(Competitive equilibrium with wedges) A competitive equilibrium with wedges is characterized by the following conditions:*

a) Individual optimality:

$$MRS_c^{ij} \leq p^j \left(1 + \tau_c^{ij}\right), \quad \forall i, \forall j \quad \textit{and} \quad MRS_n^{if} \leq w^f \left(1 + \tau_n^{if,s}\right), \quad \forall i, \forall f,$$

where the equations hold with equality when $c^{ij} > 0$ *and* $n^{if,s} > 0$*, respectively.*

b) Cost minimization and profit maximization

$$\frac{\partial G_j}{\partial x^{jk}} \leq \frac{p^k}{p^j} \frac{1 + \tau_x^{jk}}{1 + \tau_y^j}, \quad \forall j, \forall k \quad \textit{and} \quad \frac{\partial G_j}{\partial n^{jf,d}} \leq \frac{w^f}{p^j} \frac{1 + \tau^{jf}}{1 + \tau_y^j}, \quad \forall j, \forall f,$$

where the equations hold with equality when $x^{jk} > 0$ *and* $n^{jf,d} > 0$*, respectively.*

The characterization of a competitive equilibrium in this economy is standard. Individuals equalize prices or wedges to marginal rates of substitution. Firms equalize marginal products to marginal costs. We refer to economies with no wedges, $\tau_c^{ij} = \tau_n^{if,s} = \tau_x^{jk} = \tau_n^{jf,d} = \tau_y^j = 0$, as *frictionless competitive economies*. In Proposition 11, we prove a First Welfare theorem by showing that frictionless competitive economies are efficient.[17] While it should not be surprising that the First Welfare theorem applies to this economy, it is natural to provide an explicit proof since we have characterized efficiency conditions.

Proposition 11. *(First Welfare theorem) Frictionless competitive economies (*$\tau_c^{ij} = \tau_n^{if,s} = \tau_x^{jk} = \tau_n^{jf,d} = \tau_y^j = 0$*) are efficient.*

It is worth highlighting that this result relies on considering a completely frictionless environment in which all individuals have access to all markets. Any form of market segmentation would break this result. Note also that whenever $x^{jk} > 0$ and $n^{jf,d} > 0$, we can express social net valuations μ_x^{jk} and μ_n^{jf}, introduced in equations (11) and (12) as follows:

$$\mu_x^{jk} = \left(\mathbb{E}_i^{\pm}\left[1 + \tau_c^{ij}\right] \frac{1 + \tau_x^{jk}}{1 + \tau_y^j} - \mathbb{E}_i^{\pm}\left[1 + \tau_c^{ik}\right]\right) p^k \tag{30}$$

$$\mu_n^{jf} = \left(\mathbb{E}_i^{\pm}\left[1 + \tau_c^{ij}\right] \frac{1 + \tau_n^{jf,d}}{1 + \tau_y^j} - \mathbb{E}_i^{\pm}\left[1 + \tau_n^{if,s}\right]\right) w^f. \tag{31}$$

Importantly, unless goods j and k are final, and f is an elastically supplied factor, social net valuations in frictionless competitive economies will in general not be zero. This underscores the important of using network-adjusted social net valuations as the relevant source of aggregate value in disaggregated production economies. Given its importance, we state this observation in the following remark.

Remark. (Social net valuations may be non-zero in frictionless competitive economies) In disaggregated production economies, social net valuations can take non-zero values. Only in

[17] Under standard convexity assumptions, a Second Welfare theorem also trivially holds in this economy.

the case in which goods j and k are final, and f is a factor in elastic supply, social net valuations are zero.

An additional observation that emanates from the conditions in competitive economies is that marginal rates of substitution and marginal products can be read off individual and production optimality conditions. In particular, prices augmented by wedges encapsulate the relevant information to make welfare assessments. In frictionless competitive economies, efficiency follows from the fact that all individuals and producers face the same prices, equalizing their valuations at the margin.

Remark. (Observational requirements: recovering marginal rates of substitution and marginal products from prices) Optimality conditions in competitive markets encode the observational requirements necessary to compute the elements of the welfare decomposition.

Lemma 4 is helpful, because it allows us to connect the aggregate efficiency decomposition to outcomes in competitive economies. In Proposition 12, we establish several properties of the aggregate efficiency decomposition in decentralized economies.

Proposition 12. *(Properties of aggregate efficiency decomposition in decentralized economies)*

i) If final good markets are frictionless, so $\tau_c^{ij} = 0$, which implies that all individuals face the same good prices, individual consumption efficiency is zero.

ii) If factor supply markets are frictionless, so $\tau_n^{if} = 0$, which implies that all individuals face the same factor prices, then individual factor supply efficiency is zero.

iii) If final good markets and factor supply markets are frictionless, so $\tau_c^{ij} = \tau_n^{if,s} = 0$ and production markets are frictionless, so $\tau_x^{jk} = \tau_n^{jf} = \tau_y^{j}$, then production efficiency exclusively corresponds to direct technology growth.

Several important conclusions follow from Proposition 12. First, cross-individual individual efficiency exclusively depends on wedges on consumption and factor supply, regardless of the wedges on the production side of the economy. In particular, cross-sectional consumption and factor supply efficiency are given by

$$\Xi^{AE,I} = \underbrace{\sum_j p^j \mathbb{C}ov_i^{\Sigma\pm}\left[1+\tau_c^{ij}, \frac{dc^{ij}}{d\theta}\right]}_{\substack{\text{Cross-Sectional} \\ \text{Consumption Efficiency}}} - \underbrace{\sum_f w^f \mathbb{C}ov_i^{\Sigma\pm}\left[1+\tau_n^{if,s}, \frac{dn^{if}}{d\theta}\right]}_{\substack{\text{Cross-Sectional} \\ \text{Factor Supply Efficiency}}},$$

which highlights that dispersion on individual wedges is needed for cross-individual individual efficiency to be non-zero. Second, if the average wedge on individual consumption and/or factor supply is non-zero, both cross-sectional individual efficiency and production efficiency will typically be non-zero, even when cost minimization and profit optimization are frictionless. Finally, note that it is possible to have production efficiency even when good markets, factor supply markets, and cost minimization and profit optimization problem face non-zero wedges, as long as frictions exactly cancel out. This condition will generically fail to hold, so in principle any distortion will impact production efficiency.

Intermediates inverse matrix in market economies. Finally, we would like to explain how the intermediates inverse matrix introduced in Section 5 can be expressed in terms of a conventional Leontief Inverse matrix in frictionless economies and in terms of a cost-based Leontief Inverse matrix in economies with frictions.

Proposition 13. *(Intermediates inverse matrix in decentralized economies)*

a) In frictionless competitive economies, the matrix $\left(I - \hat{\boldsymbol{y}}^{-1}\boldsymbol{G}_x\boldsymbol{\xi}\hat{\boldsymbol{y}}\right)^{-1}$ becomes the (standard, revenue-based) Leontief Inverse.

b) In competitive economies with wedges, when there is only one wedge per producer (i.e., a markup) and technologies satisfy constant returns to scale, the matrix $\left(I - \hat{\boldsymbol{y}}^{-1}\boldsymbol{G}_x\boldsymbol{\xi}\hat{\boldsymbol{y}}\right)^{-1}$ becomes the cost-based Leontief Inverse introduced in Baqaee and Farhi (2020).

As shown in Section 5.3, the intermediates inverse matrix can be expressed in terms of the matrix of log-marginal products. Proposition 13 such that the matrix of log-marginal products becomes a revenue- or cost-based Leontief inverse. This result explains why Leontief Inverse matrices are ubiquitous in the study of the aggregate impact of shocks in disaggregated economies — see e.g. Liu (2019) or Baqaee and Farhi (2020). This is a non-obvious result that deserves further study.

8 Hulten's Theorem Revisited

Hulten's theorem (Hulten, 1978) has played a prominent role in the literatures on the macroeconomic impact of microeconomic shocks and growth accounting; see among others, Gabaix (2011), Carvalho and Tahbaz-Salehi (2019), or Baqaee and Farhi (2020). The standard Hulten's theorem, which we refer to as *"Output Hulten's theorem"* in this paper , is typically stated as:

> *"In efficient economies, the marginal impact on output of a Hicks-neutral technology shock to a given industry is equal to its Domar weight (sales as a share of aggregate output)".*

Hence, Output Hulten's theorem as typically stated is a result that applies to the impact of technology shocks on output in the case of efficient economies. In this section, we introduce a new Welfare Hulten's theorem that applies to the impact of technology shocks on welfare.

In the following theorem, we present a result that characterizes the impact of Hicks-neutral technology shocks on the aggregate efficiency component of welfare assessments for frictionless competitive economies with heterogeneous agents, elastic factor supplies, arbitrary preferences and technologies, and arbitrary social welfare functions.

Theorem 1. *(Welfare Hulten's theorem) In frictionless competitive economies, the aggregate efficiency component of welfare assessments associated with a proportional Hicks-neutral technology change in technology j, which we denote here by $\Xi^{AE,P,G}$, is exclusively given by the technology growth component of production efficiency, which can be expressed as*

$$\frac{\Xi^{AE,P,G}}{\sum_j p^j c^j} = \underbrace{\frac{p^j y^j}{\sum_j p^j c^j}}_{=Domar\ Weight},$$

where $\frac{p^j y^j}{\sum_j p^j c^j}$ corresponds to the Domar weight.

Consequently, Domar weights are indeed the relevant variable to capture the marginal impact of technology shocks from a welfare perspective. We would like to make several observations related to the Welfare Hulten's theorem.

First, when there is a single individual ($I = 1$), redistribution is zero, so aggregate efficiency equals welfare. In that case, the Domar weight exactly captures the welfare change, not only the aggregate efficiency component.

Second, it is well understood that Output Hulten's theorem does not apply to economies with elastic factor supplies. For instance, Baqaee and Farhi (2018) state that *"Hulten's theorem fails when factors supplies are elastic".* The reason why Welfare Hulten's theorem applies to economies with elastic factors is that welfare assessments account for the cost of supplying factors — the aggregate factor supply efficiency component in Equation (9). This contrast between Output and Welfare Hulten's theorem highlights that Hulten's theorem is at its core a result about welfare, not about output. However, when $I = 1$ and factors are supplied inelastically, both notions coincide, which has justified the use of Hulten's theorem as a result about output.

Third, note that we state that the Welfare Hulten's theorem applies to "frictionless competitive" economies, not to "efficient" economies.[18] This is a subtle but relevant distinction. In any economy considered in the paper, based on Proposition 6, the technology growth component of aggregate efficiency component of welfare assessments associated with a proportional Hicks-neutral technology change in technology j is given by[19]

$$\Xi^{AE,P,G} = \left(\boldsymbol{\mu}_x' \left(\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x\right)^{-1} \boldsymbol{\xi} + \boldsymbol{MRS}_c'\right) \boldsymbol{y}.$$

The content of the Welfare Hulten's theorem is the observation that two things occurs in

[18]The same observation applies to the standard Output Hulten's theorem, which applies to frictionless competitive economies, not to efficient economies.

[19]Note that for proportional Hicks-neutral technology shocks, it is the case that $\boldsymbol{G}_\theta = \boldsymbol{y}$.

frictionless competitive economies: i) the allocative efficiency components of aggregate efficiency are zero, and ii) the following condition holds:

$$\boldsymbol{\mu}_x' \left(\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x\right)^{-1} \boldsymbol{\xi} + \boldsymbol{MRS}_c' = \boldsymbol{p}^T, \tag{32}$$

where $\boldsymbol{p}$ denotes a $J \times 1$ vector of good prices. When an economy is efficient, the allocative efficiency components are necessarily zero, so $\Xi^{AE} = \Xi^{AE,P,G}$. However, efficiency does not guarantee that Equation (32) holds. That is, there may be efficient allocations in which Equation (32) does not hold — the minimal vertical economy in Section 10 is an example. However, Equation (32) does hold in frictionless competitive economies, guaranteeing that Welfare Hulten's theorem holds. Importantly, in interior economies efficiency in the use of intermediates requires immediately implies that $\boldsymbol{\mu}_x' \left(\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x\right)^{-1} = 0$. In that case, it is evident that Equation (32) holds, since $\boldsymbol{MRS}_c' = \boldsymbol{p}^T$ in an interior frictionless competitive economy. However, when there are pure intermediate goods, efficiency in the use of intermediates does not imply that $\boldsymbol{\mu}_x' \left(\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x\right)^{-1} = 0$ or that $\boldsymbol{MRS}_c' = \boldsymbol{p}^T$ (remember that the components of $\boldsymbol{MRS}_c'$ associated with pure intermediates will be zero). Remarkably, in those cases, competitive forces guarantee that Equation (32), and the Welfare Hulten's theorem holds.

Fourth, it should be evident that Domar weights appear because we have chosen to normalize aggregate efficiency by $\sum_j p^j c^j$, which is a measure of nominal final output. In general, Hulten's theorem simply implies that for proportional Hicks-neutral technology shocks, it must be that, in matrix from, when allowing for technology changes to all technologies:

$$\Xi^{AE,P,G} = \boldsymbol{p}^T \boldsymbol{y} = \sum_j p^j y^j.$$

9 Redistribution

While we have so far focused on understanding the aggregate efficiency component associated with a particular perturbation, it is worth highlighting the determinants of the redistribution component in Proposition 1. Besides the individual weights $\tilde{\omega}^i$, which can be interpreted as direct choices for a planner (see Dávila and Schaab (2022) for this interpretation), the only input to compute the redistribution component is the change in individual utility expressed in units of the welfare numeraire, $\frac{\frac{dV_i}{d\theta}}{\lambda_i}$. In Proposition 14, we provide two different characterizations of $\frac{\frac{dV_i}{d\theta}}{\lambda_i}$ and study their properties.

Proposition 14. *(Redistribution)*

a) The change in individual utility expressed in units of the welfare numeraire is given by

$$\frac{\frac{dV_i}{d\theta}}{\lambda_i} = \sum_j MRS_c^{ij} \frac{dc^{ij}}{d\theta} - \sum_f MRS_n^{if} \frac{dn^{if,s}}{d\theta}. \tag{33}$$

b) In competitive economies with wedges, the change in individual utility expressed in units of the welfare numeraire can be expressed as

$$\frac{\frac{dV_i}{d\theta}}{\lambda_i} = \underbrace{-\sum_j \frac{dp^j}{d\theta} c^{ij} + \sum_f \frac{dw^f}{d\theta} n^{if,s} + \sum_j \zeta^{ij} \frac{d\pi^j}{d\theta}}_{\text{Distributive Pecuniary Effects}} + \underbrace{\sum_j p^j \tau^{p,ij} \frac{dc^{ij}}{d\theta} - \sum_f w^f \tau^{w,if} \frac{dn^{if}}{d\theta}}_{\text{Distortionary Effects}}, \quad (34)$$

where distributive pecuniary effects add up to zero in the absence of technology or endowment growth.[20]

Part a) of Proposition 14 highlights that in order to understand changes in individual welfare it is sufficient to understand changes in individual consumption and factor supply. Unfortunately, without any additional structure, it is impossible to understand further how $\frac{dc^{ij}}{d\theta}$ and $\frac{dn^{if,s}}{d\theta}$ change in a given perturbation. However, in competitive economies with or without wedges, it is possible to provide a different formulation of $\frac{\frac{dV_i}{d\theta}}{\lambda_i}$. This new formulation, introduced in Proposition 14) exploits the duality between changes in prices and changes in quantities that needs to be satisfied when there are budget constraints. Equation (34) shows that individual utility changes can be traced to changes in prices (and profits), which correspond to distributive pecuniary effects — see Dávila and Korinek (2018) — and are present in any competitive economy, and changes in allocations distorted by wedges, which refer to as distortionary effects.

Hence, the redistribution component can be mapped to changes in prices and changes in allocations distorted by wedges. Therefore, a policy that benefits those individual with higher distortions will be valuable if a planner gives more weight to those individuals, perhaps because they are poorer. But in efficient economies, the redistribution component exclusively captures the distributive pecuniary effects of policies.

It is worth highlighting that the sum of distributive pecuniary effects is zero in the absence of technology growth or factor endowment growth. In frictionless competitive economies, Welfare Hulten's Theorem implies that $\sum_i \frac{\frac{dV_i}{d\theta}}{\lambda_i} = \sum_j p^j \frac{\partial G^j}{\partial \theta}$. In that sense, the well-known results that distributive pecuniary effects are zero-sum in static competitive economies, see e.g, Dávila and Korinek (2018), does not hold when there is technology growth.

10 Applications

Before concluding, we systematically present the minimal applications that feature each of the six allocative efficiency components of the decomposition of aggregate efficiency introduced in this paper. We also introduce a new economy, which we refer to as the minimal general production economy, which is the simplest economy that captures all welfare-relevant phenomena

[20]This formulation assume that all wedges are purely allocational, so there are is no physical waste of resources associated with the existence of wedges. It is straightforward to consider the alternative case in which wedges involve a direct deadweight loss.

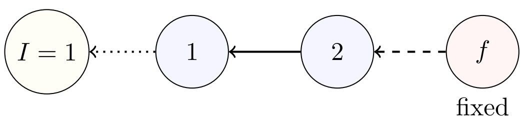
	Cross-Sectional Individual Efficiency		Production Efficiency			
	Cross-Sectional Consumption Efficiency	Cross-Sectional Factor Supply Efficiency	Cross-Sectional Intermediate Input Efficiency	Aggregate Intermediate Input Efficiency	Cross-Sectional Factor Use Efficiency	Aggregate Factor Use Efficiency
Minimal Vertical	×	×	×	×	×	×
Minimal General Production	×	×	✓	✓	✓	✓
Robinson Crusoe	×	×	×	×	×	✓
Minimal Horizontal	×	×	×	×	✓	×
Minimal Roundabout	×	×	×	✓	×	×
Minimal Diversified Intermediate	×	×	✓	×	×	×
Minimal Two-Factor Suppliers	×	✓	×	×	×	✓
Edgeworth Box	✓	×	×	×	×	×

Table 1: Summary of Applications

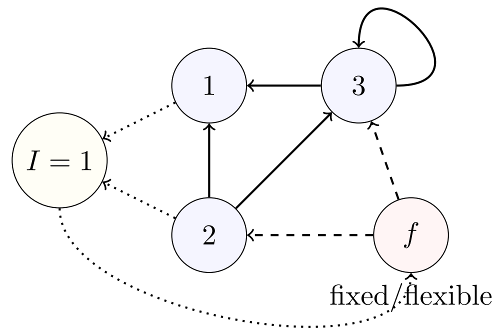
Note: This table illustrates the components of the welfare decomposition that can possible be non-zero. The minimal vertical economy is efficient by construction, so welfare assessments in that economy can only increase or decrease efficiency via technology growth.

in disaggregated production economies. We hope that this economy becomes the workhorse for future analysis of disaggregated production networks. Our first application corresponds to a minimal vertical economy that is trivially efficient. In this economy, no allocative efficiency gains/losses are possible, so production efficiency can only be due to the direct technology change component.

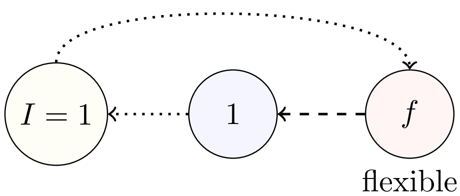
For each application, we focus on characterizing the aggregate efficiency component in response to a technology change in the production of all goods, unless explicitly stated otherwise. Figure 2 illustrates the technological possibilities associated with all eight applications. Table 1 summarizes the components through which perturbations may impact welfare in each economy.



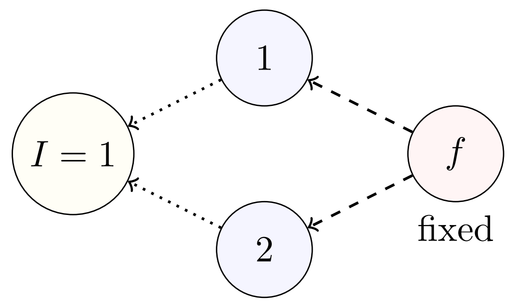
(a) Minimal Vertical Economy



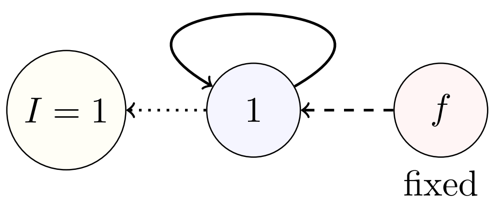
(b) Minimal General Production Economy



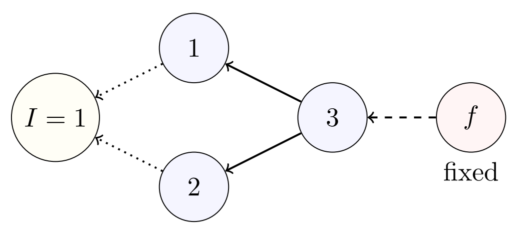
(c) Robinson Crusoe Economy



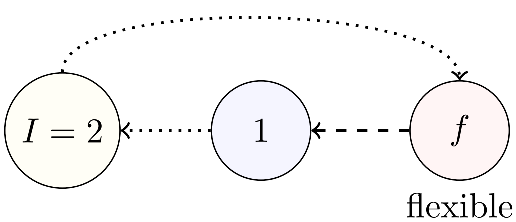
(d) Minimal Horizontal Economy



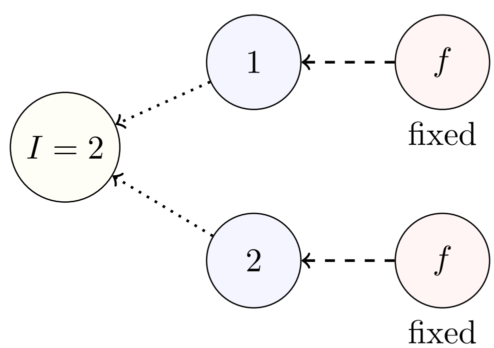
(e) Minimal Roundabout Economy



(f) Minimal Diversified Intermediate



(g) Minimal Two-Factor Suppliers



(h) Edgeworth Box Economy

Figure 2: Illustrative Applications

Note: This figure represents the technology structure associated with each of the illustrative applications. The first application corresponds to a minimal vertical economy, which is always efficient. The second application corresponds to the minimal general economy, which is the minimal economy that features all forms of production efficiency. The remaining applications correspond to the minimal environments in which each of the six allocative efficiency components of the decomposition of aggregate efficiency introduced in Proposition 6 are the single component of production efficiency.

1. Minimal Vertical Economy ($I = 1,\ J = 2,\ F = 1$) The minimal vertical economy is the simplest economy that illustrates the role played by pure intermediate goods. In this economy, there is single final good produced using a single intermediate good, which is in turn produced by a single factor in fixed supply. Formally, using the notation introduced in Section 2, this economy can be formalized as

$$\begin{aligned} V_1 &= u\left(c^{11}\right), \quad & y^1 &= G^1\left(x^{12};\theta\right), \text{ and} \quad & y^2 &= G^2\left(n^{21,d};\theta\right) \\ y^1 &= c^{11}, & y^2 &= x^{12}, & \text{and} \quad \bar{n}^{1,s}(\theta) &= n^{21,d}. \end{aligned}$$

This economy has a unique feasible allocation and is trivially efficient, since there are no feasible perturbations.[21] In this economy, we can express production efficiency, which is exclusively given by technology growth, as

$$\begin{aligned} \Xi^{AE,P} &= \psi_x^{12}\frac{dx^{12}}{d\theta} + MRS_c^{11}\frac{\partial G^1}{\partial\theta} \\ &= \underbrace{MRS_c^{11}\frac{\partial G^1}{\partial x^{12}}}_{=\psi_G^2}\frac{\partial G^2}{\partial\theta} + \underbrace{MRS_c^{11}}_{=\psi_G^1}\frac{\partial G^1}{\partial\theta} \\ &= p^2\frac{\partial G^2}{\partial\theta} + p^1\frac{\partial G^1}{\partial\theta}, \end{aligned}$$

where our last line uses the fact that $MRS_c^{11}\frac{\partial G^1}{\partial x^{12}} = p^1\frac{p^2}{p^1} = p^2$ in the frictionless competitive case. In this case, note that social net valuations and network adjusted social net valuations coincide, since $\psi_x^{12} = \mu_x^{12} = MRS_c^{11}\frac{\partial G^1}{\partial x^{12}}$, and that there is no need for intermediate inverse matrix, since $\frac{dx^{12}}{d\theta} = \frac{\partial G^2}{\partial\theta}$. This derivation illustrates how both Welfare and Output Hulten's theorems require a frictionless competitive environment, so $\frac{\partial G^1}{\partial x^{12}} = \frac{p^2}{p^1}$. Any wedge in this condition would make Hulten's theorem to fail even though this economy is always efficient.

2. Minimal General Production Economy ($I = 1,\ J = 3,\ F = 1$) In this paper, we introduce a new economy that acts as the counterpart of the Edgeworth Box for production economies, in the sense that it is the simplest economy that allows us to study all relevant welfare phenomena in disaggregated production economies. Using the notation introduced in

[21] This economy illustrates how to accommodate transportation/iceberg costs. If good 1 is interpreted as good-at-destination, while good 2 is interpreted as good-at-origin, transportation/iceberg costs can be modeled by assuming that $G^1(\cdot)$ is given by $y^1 = \left(1 - \tau^{12}\right)x^{12}$, where τ^{12} can plausibly increase with the "distance" between origin and destination.

Section 2, a minimal general production economy can be expressed as

$$
\begin{aligned}
V_1 = u\left(c^{11}, c^{12}\right) \quad &\text{and} \quad y^1 = G^1\left(x^{13}, x^{12}; \theta\right) \\
y^2 = G^2\left(n^{21,d}; \theta\right) \quad &\text{and} \quad y^3 = G^3\left(x^{32}, x^{33}, n^{31,d}; \theta\right) \\
y^1 = c^{11} \quad &\text{and} \quad y^2 = c^{12} + x^{12} + x^{32} \\
y^3 = x^{13} + x^{33} \quad &\text{and} \quad \bar{n}^{1,s}(\theta) = n^{21,d} + n^{31,d}.
\end{aligned}
$$

Regarding intermediate inputs, this economy is general in the sense that it simultaneously features the three types of goods: one pure final good (good 1), one mixed good (good 2), and one pure intermediate (good 3). Moreover, it simultaneously features the four possible links between goods: i) it links a pure intermediate to a pure intermediate ($3 \leftarrow 3$); ii) it links a pure intermediate to a final ($1 \leftarrow 3$); iii) it links a final (mixed) good to a pure intermediate ($3 \leftarrow 2$); and iv) it links a final (mixed) to a final good. Moreover, when $x^{33} = 0$, the pure intermediate good is specialized, while when $x^{33} > 0$, the pure intermediate good is diversified. Regarding factors, this economy is general in the sense that it links its factor to both final (good 2) and intermediate (good 3) goods. Moreover, when $n^{21,d} = 0$, the factor is specialized, while when $n^{21,d} > 0$ the factor is diversified.

Here, for simplicity, we assume that $n^{21,d} = 0$, so the single factor is specialized on good 3. We also assume that factor endowment is fixed and we exclusively consider a change in the technology of good 3, which is the pure intermediate. The welfare decomposition in levels takes the form

$$
\Xi^{AE,P} = \mu^{12}\frac{dx^{12}}{d\theta} + \mu^{13}\frac{dx^{13}}{d\theta} + \mu^{32}\frac{dx^{32}}{d\theta} + \mu^{33}\frac{dx^{33}}{d\theta},
$$

where social net valuations are given by

$$
\begin{aligned}
\mu^{12} &= \mathbb{E}_i^{\pm}\left[MRS^{11}\right]\frac{\partial G^1}{\partial x^{12}} - \mathbb{E}_i^{\pm}\left[MRS^{12}\right] \quad &&\text{and} \quad \mu^{32} = -\mathbb{E}_i^{\pm}\left[MRS^{12}\right] \\
\mu^{13} &= \mathbb{E}_i^{\pm}\left[MRS^{11}\right]\frac{\partial G^1}{\partial x^{13}} &&\text{and} \quad \mu^{33} = 0.
\end{aligned}
$$

As highlighted in the text, $\mu^{33} = 0$, since it corresponds to a pure intermediate good used in the production of another pure intermediate good. Also, $\mu^{32} < 0$ and $\mu^{13} > 0$, which highlights the fact that social net valuations are typically non-zero, regardless of the competitive structure of the economy. In frictionless competitive economies, $\mu^{12} = 0$, but in general $\mu^{12} \gtreqless 0$.

The equations that correspond to the intermediate inverse matrix are

$$
\begin{aligned}
\frac{dx^{12}}{d\theta} &= \frac{d\xi^{12}}{d\theta}y^2 \qquad \frac{dx^{13}}{d\theta} = \frac{d\xi^{13}}{d\theta}y^3 + \xi^{13}\left(\frac{\partial G^3}{\partial x^{32}}\frac{dx^{32}}{d\theta} + \frac{\partial G^3}{\partial x^{33}}\frac{dx^{33}}{d\theta} + \frac{\partial G^3}{\partial \theta}\right) \\
\frac{dx^{32}}{d\theta} &= \frac{d\xi^{32}}{d\theta}y^2 \qquad \frac{dx^{33}}{d\theta} = \frac{d\xi^{33}}{d\theta}y^3 + \xi^{33}\left(\frac{\partial G^3}{\partial x^{32}}\frac{dx^{32}}{d\theta} + \frac{\partial G^3}{\partial x^{33}}\frac{dx^{33}}{d\theta} + \frac{\partial G^3}{\partial \theta}\right),
\end{aligned}
$$

which we solve in the appendix for $\frac{dx^{13}}{d\theta}$ — since $\mu^{33}=0$, $\frac{dx^{33}}{d\theta}$ does not directly impact the welfare calculations. In the frictionless competitive case, since $\mu^{11}=0$, we can express aggregate/production efficiency as

$$\Xi^{AE,P}=\mu^{13}\frac{dx^{13}}{d\theta}+\mu^{32}\frac{dx^{32}}{d\theta}=p^{3}\frac{dx^{13}}{d\theta}-p^{2}\frac{dx^{32}}{d\theta}=p^{3}\frac{\partial G^{3}}{\partial\theta},$$

where the last equality follows a serious of non-trivial cancellations that we derive in the appendix. This application illustrates the power of competitive markets aligning decisions to deliver an efficient outcome and generate the Welfare Hulten's theorem.

3. Robinson Crusoe Economy ($I=1$, $J=1$, $F=1$) One-producer one-consumer economies (i.e., Robinson Crusoe economies) are the simplest to study production (see Section 15.C of Mas-Colell, Whinston and Green (1995)). In these economies, a single individual consumes a single good and elastically supplies a single factor of production. A single production technology uses the supplied factor to produce the good. Formally, using the notation introduced in Section 2, preferences, technology, and resource constraints are given by

$$V_1=u\left(c^{11},n^{11,s}\right)\quad\text{and}\quad y^{1}=G^{1}\left(n^{11,d};\theta\right)$$
$$y^{1}=c^{11}\quad\text{and}\quad n^{11,s}=n^{11,d}.$$

This economy feature no intermediate inputs, and is useful to compare the case of elastic and inelastic factors. In this case, we can express aggregate/production efficiency as

$$\begin{aligned}\Xi^{AE,P}&=\mu_{n}^{1}\frac{dn^{11,d}}{d\theta}+MRS_{c}^{11}\frac{\partial G^{1}}{\partial\theta}\\&=\underbrace{\left(MRS_{c}^{11}\frac{\partial G^{1}}{\partial n^{11}}-MRS_{n}^{11}\right)}_{=\overline{\psi}_{n}^{1}}\frac{dn^{1,s}}{d\theta}+\underbrace{MRS_{c}^{11}}_{=\psi_{G}^{1}}\frac{\partial G^{1}}{\partial\theta}\\&=p^{1}\frac{\partial G^{1}}{\partial\theta},\end{aligned}$$

where our last line uses the fact that $MRS_{c}^{11}\frac{\partial G^{1}}{\partial n^{11}}-MRS_{n}^{11}=p^{1}\frac{\partial G^{1}}{\partial n^{11}}-w^{1}=0$ in the frictionless competitive case. This economy illustrates that aggregate factor efficiency often captures *labor wedges,* which are typically present in economies with rigid prices and/or labor market frictions.

4. Minimal Horizontal Economy ($I=1$, $J=2$, $F=1$) The minimal horizontal economy is the simplest to illustrate the role played by the possibility of reallocating factors across different uses. This economy generalizes to many well-known frameworks, including Hecksher-Ohlin, Armington (1969), and Hsieh and Klenow (2009). In these economies, a single individual consumes two different goods that can be produced using the same factor, which we assume to be in fixed supply. Formally, using the notation introduced in Section 2, preferences, technology, and resource constraints are given by

$$
\begin{aligned}
V_1 = u\left(c^{11}, c^{12}\right), \quad & y^1 = G^1\left(n^{11,d}; \theta\right), \text{ and } \quad & y^2 = G^2\left(n^{21,d}; \theta\right) \\
y^1 = c^{11}, \quad & y^2 = c^{12}, \quad & \text{and} \quad \bar{n}^{1,s}(\theta) = n^{11,d} + n^{21,d}.
\end{aligned}
$$

Production efficiency can only be due to cross-sectional factor use efficiency and technology growth, since this economy features no intermediate inputs and its single factor is in fixed supply. We can express production efficiency as

$$
\begin{aligned}
\Xi^{AE,P} &= \mathbb{C}ov_j^{\Sigma}\left[\psi_n^{j1}, \frac{d\chi^{j1,d}}{d\theta}\right] n^{1,d} + \underbrace{MRS_c^{12}}_{=\psi_G^2} \frac{\partial G^2}{\partial \theta} + \underbrace{MRS_c^{11}}_{=\psi_G^1} \frac{\partial G^1}{\partial \theta} \\
&= p^2 \frac{\partial G^2}{\partial \theta} + p^1 \frac{\partial G^1}{\partial \theta}
\end{aligned}
$$

where the last line applies in frictionless competitive markets, since $\psi_n^{11} = \psi_n^{21}$ in that case. In general, note that $\frac{d\chi^{11,d}}{d\theta} + \frac{d\chi^{21,d}}{d\theta} = 0$, and that social net valuations (no need for network-adjustments in this case) are given by

$$
\psi_n^{11} = MRS_c^{11} \frac{\partial G^1}{\partial n^{11,d}} - MRS_n^{11} \quad \text{and} \quad \psi_n^{21} = MRS_c^{i2} \frac{\partial G^2}{\partial n^{21,d}} - MRS_n^{11}.
$$

This application illustrates the results in Proposition 10, in which efficiency requires equalization of marginal products of a given factor in units of the numeraire:

$$
MRS_c^{11} \frac{\partial G^1}{\partial n^{11,d}} = MRS_c^{12} \frac{\partial G^2}{\partial n^{21,d}}.
$$

In this case, if goods were perfect substitutes, $MRS_c^{11} = MRS_c^{12}$, which implies that marginal product equalization is needed for efficiency, that is, $\frac{\partial G^1}{\partial n^{11,d}} = \frac{\partial G^2}{\partial n^{21,d}}$.

5. Minimal Roundabout Economy ($I = 1$, $J = 1$, $F = 1$) Roundabout economies have been used to illustrate how intermediate goods impact production — see e.g., Jones (2011). The minimal roundabout economy is the simplest economy in which aggregate intermediate input efficiency can exist. In this economy a single individual consumers a single mixed good, which is at the same time final and intermediate to itself. Formally, using the notation introduced in Section 2, preferences, technology, and resource constraints are given by

$$
\begin{aligned}
V_1 = u\left(c^{11}\right) \quad & \text{and} \quad y^1 = G^1\left(x^{11}, n^{11,d}; \theta\right) \\
y^1 = c^{11} + x^{11} \quad & \text{and} \quad \bar{n}^{1,s}(\theta) = n^{11,d}.
\end{aligned}
$$

In this case, the social net valuation of changing x^{11} is given by $\mu^{11} = MRS_c^{11} \frac{\partial G^1}{\partial x^{11}} - MRS_c^{11}$ and the network adjusted social net valuation corresponds to

$$\Xi^{AE,P} = \mu^{11} \frac{dx^{11}}{d\theta} + MRS_c^{11} \frac{\partial G^1}{\partial \theta},$$
$$= \frac{MRS_c^{11} \left(\frac{\partial G^1}{\partial x^{11}} - 1 \right)}{1 - \xi^{11} \frac{\partial G^1}{\partial x^{11}}} \left(\frac{d\xi^{11}}{d\theta} y^1 + \xi^{11} \frac{\partial G^1}{\partial \theta} \right) + MRS_c^{11} \frac{\partial G^1}{\partial \theta},$$

where we use the intermediate inverse matrix, which in this case is given by $\left(1 - \xi^{11} \frac{\partial G^1}{\partial x^{11}}\right)^{-1}$. We can therefore express aggregate/production efficiency as

$$\Xi^{AE,P} = \overline{\psi}_x^1 \frac{d\phi^1}{d\theta} y^1 + \psi_G^1 \frac{\partial G^1}{\partial \theta} = p^1 \frac{\partial G^1}{\partial \theta},$$

where the last equality is valid in frictionless competitive economies. The minimal roundabout economy highlights the role played by the intermediate inverse matrix and the difference between social net valuations and network-adjusted social net valuations.

6. Minimal Diversified Intermediate ($I = 1$, $J = 3$, $F = 1$) The minimal diversified intermediate economy is the simplest economy in which cross-sectional intermediate input efficiency can exist. In this economy, a single individual consumes two final goods, that are exclusively produced using a third, which is a pure intermediate. This pure intermediate is produced using a single factor in fixed supply. Formally, using the notation introduced in Section 2, preferences, technology, and resource constraints are given by

$$V_1 = u\left(c^{11}, c^{12}\right), \quad y^1 = G^1\left(x^{13}; \theta\right), \quad y^3 = G^3\left(n^{31,d}; \theta\right) \text{ and } \quad y^2 = G^2\left(x^{23}; \theta\right)$$
$$y^1 = c^{11}, \quad y^2 = c^{12}, \quad y^3 = x^{13} + x^{23} \quad \text{and} \quad \bar{n}^{1,s} = n^{31,d}.$$

For simplicity, in this case we only consider technology changes in the pure intermediate good 3. In that case, we can express aggregate/production efficiency as

$$\Xi^{AE,P} = \mu^{13} dx^{13} + \mu^{23} dx^{23},$$

where the inverse intermediate matrix is given by the solution to

$$\frac{dx^{13}}{d\theta} = \frac{d\xi^{13}}{d\theta} y^3 + \xi^{13} \frac{\partial G^3}{\partial \theta}$$
$$\frac{dx^{23}}{d\theta} = \frac{d\xi^{23}}{d\theta} y^3 + \xi^{23} \frac{\partial G^3}{\partial \theta},$$

and where $\psi_x^{13} = \mu_x^{13} = MRS_c^{11} \frac{\partial G^1}{\partial x^{13}}$ and $\psi_x^{23} = \mu_x^{23} = MRS_c^{12} \frac{\partial G^2}{\partial x^{23}}$. In this case, $\phi^3 = 1$ and

$\phi^1 = \phi^2$, so the only source of production can be cross-sectional intermediate input efficiency, so

$$\begin{aligned}\Xi^{AE,P} &= \mathbb{C}ov_j^{\Sigma}\left[\psi_x^{jk}, \frac{d\chi_x^{jk}}{d\theta}\right] x^k + \left(MRS_c^{11}\frac{\partial G^1}{\partial x^{13}}\xi^{13} + MRS_c^{12}\frac{\partial G^2}{\partial x^{23}}\xi^{23}\right)\frac{\partial G^3}{\partial \theta} \\ &= p^3\frac{\partial G^3}{\partial \theta},\end{aligned}$$

where the last equality is valid in frictionless competitive economies, since $\xi^{13} + \xi^{23} = 0$, $MRS_c^{11}\frac{\partial G^1}{\partial x^{13}} = p^1\frac{p^3}{p^1} = p^3$ and $MRS_c^{12}\frac{\partial G^2}{\partial x^{23}} = p^2\frac{p^3}{p^2} = p^3$.

7. Minimal Two Factor Supplier Economy ($I = 2$, $J = 1$, $F = 1$) The minimal two factor supplier economy (we could also called it Robinson Crusoe and Friday economy) is the minimal economy in which perturbations can change cross-sectional factor supply efficiency. In this economy, we assume that two individuals have identical linear preferences for consumption of a single produced good, which we use as numeraire. This eliminates potential gains from cross-sectional consumption efficiency, since $MRS_n^{11} = MRS_c^{21} = 1$. We also assume that there is a single production technology that uses a single factor that can be supplied either of the two individuals, with in principle different disutility. Formally, using the notation introduced in Section 2, preferences, technology, and resource constraints are given by

$$\begin{aligned}V_1 = c^{11} + u_1\left(n^{11,s}\right) \quad &\text{and} \quad V_2 = c^{21} + u_2\left(n^{21,s}\right) \\ y^1 = G^1\left(n^{11,d};\theta\right) \qquad &y^1 = c^{11} + c^{21} \\ n^{11,s} + n^{21,s} &= n^{11,d}.\end{aligned}$$

This economy is designed to illustrate the possibility of cross-sectional individual factor supply efficiency, so we abstract from technology changes. In this case, we can write cross-sectional individual efficiency as

$$\Xi^{AE,I} = \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_n^{i1}, \frac{dn^{i1,s}}{d\theta}\right],$$

where $MRS_n^{i1} = \frac{u_i'\left(n^{11,s}\right)}{\lambda^i}$ and where $\Xi^{AE,I} = 0$ in frictionless competitive economies, since in that case $MRS_n^{i1} = \overline{MRS}_n^1$. Finally, note that in this economy, production efficiency features an aggregate factor use component, which is generically unavoidable in environments with flexible factor supply. Production in this case is given by

$$\begin{aligned}\Xi^{AE,P} &= \underbrace{\overbrace{\mathbb{E}_i^{\pm}\left[MRS_c^{i1}\right]}^{=1}\frac{\partial G^1}{\partial n^{11,d}} - \mathbb{E}_i^{\pm}\left[MRS_n^{i1}\right]}_{=\overline{\psi}_n^1}\frac{dn^{1,s}}{d\theta} \\ &= \left(w^1 - \mathbb{E}_i^{\pm}\left[MRS_n^{i1}\right]\right)\frac{dn^{1,s}}{d\theta} = 0,\end{aligned}$$

where the last two equalities are valid in frictionless competitive economies, since $\frac{\partial G^1}{\partial n^{11,d}} = w^1$ and $\mathbb{E}_i^{\pm}\left[MRS_n^{i1}\right] = w^1$ using consumption as numeraire.

8. Edgeworth Box Economy ($I = 2$, $J = 2$, $F = 2$) Pure exchange economies (i.e., Edgeworth Box economies) are the simplest to study most phenomena in general equilibrium and welfare economics. In this economy two individuals consume two different goods, which appear as endowments. By assuming that each good is produced with a single factor in fixed supply, we illustrate how our environment nests pure endowment economies. Formally, using the notation introduced in Section 2, an Edgeworth Box economy can be expressed as

$$\begin{aligned} V_1 = u\left(c^{11}, c^{12}\right) \quad &\text{and} \quad V_2 = u\left(c^{21}, c^{22}\right) \\ y^1 = G^1\left(n^{11,d}; \theta\right) \quad &\text{and} \quad y^2 = G^2\left(n^{22,d}; \theta\right) \\ y^1 = c^{11} + c^{21} \quad &\text{and} \quad y^2 = c^{12} + c^{22} \\ \bar{n}^{1,s}\left(\theta\right) = n^{11,d} \quad &\text{and} \quad \bar{n}^{2,s}\left(\theta\right) = n^{22,d}. \end{aligned}$$

This version of an Edgeworth Box economy highlights that changes in the endowments of goods can be interpreted as changes in technologies. This Edgeworth Box economy is designed to explain the role individual heterogeneity in consumption.[22] In this economy, changes in production efficiency are simply given by

$$\Xi^{AE,P} = \underbrace{MRS_c^{11}}_{=\psi_G^1} \frac{\partial G^1}{\partial \theta} = p^1 \frac{\partial G^1}{\partial \theta},$$

which is another trivially manifestation of Welfare Hulten's theorem. Changes in cross-sectional consumption efficiency are given by

$$\Xi^{AE,I} = \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{i1}, \frac{dc^{i1}}{d\theta}\right] + \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{i2}, \frac{dc^{i2}}{d\theta}\right],$$

where $\Xi^{AE,I} = 0$ in frictionless competitive economies, since in that case $MRS_c^{i1} = \overline{MRS}_c^1$ and $MRS_c^{i2} = \overline{MRS}_c^2$.

[22] It is important to highlight that if the true economy is an endowment economy, in which the primitives are endowments of a particular good, like a textbook Edgeworth Box economy, subdecomposing production efficiency is meaningless. In other words, one can construct different production economies that map to the same endowment economy but that will attribute welfare assessments to different elements of the decomposition of production efficiency. Subject to this observation, it is without loss of generality to think of changes in endowments as changes in technology growth.

11 Conclusion

This paper introduces a welfare accounting framework in general economies with heterogeneous agents and disaggregated production technologies. Our welfare decomposition expands our understanding of the ultimate origins of welfare changes and allows us to provide the first general characterization of efficiency conditions for disaggregated production economies. The Welfare Hulten's theorem that we introduce demonstrates the critical role of accounting for the welfare cost of supplying elastic factors and shows that Hulten's theorem is fundamentally a result about welfare, rather than output. Our results underscore the importance of properly accounting for non-negativity constraints in feasible allocations, especially in the context of sparse production networks and diverse consumption patterns among individuals.

By examining the properties of the welfare decomposition in decentralized economies, we have shown how prices can be instrumental in recovering the objects needed for welfare assessments: marginal rates of substitutions and marginal products. Additionally, our analysis shows that distributive pecuniary and distortionary effects matter for redistribution. Finally, the paper illustrates our results in a range of minimal economy models, including the minimal general production economy, which has the potential to become the workhorse for the study of welfare in disaggregated production economies.

In ongoing work, we extend the results of this paper to dynamic stochastic economies with incomplete markets, in which new considerations associated with the presence of investment, risk-sharing, and intertemporal sharing need to be independently studied.

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Appendix

A Matrix Definitions

In this section we formally define all the matrices used to present our results.

Allocations. We collect output of all goods, y^j, in a $J \times 1$ vector $\boldsymbol{y}$ and a $J \times J$ diagonal matrix $\hat{\boldsymbol{y}}$:

$$\boldsymbol{y} = \begin{pmatrix} y^1 \\ \vdots \\ y^J \end{pmatrix}_{J\times 1} \quad \text{and} \quad \hat{\boldsymbol{y}} = \begin{pmatrix} y^1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & y^J \end{pmatrix}_{J\times J}.$$

We also collect intermediate-output shares $\xi^{jk} = \frac{x^{jk}}{y^k}$ in the $JK \times J$ matrix $\boldsymbol{\xi}$, as follows:

$$\boldsymbol{\xi} = \begin{pmatrix} \boldsymbol{\xi}^1 \\ \vdots \\ \boldsymbol{\xi}^K \end{pmatrix}_{JK\times K} \quad \text{where} \quad \boldsymbol{\xi}^1 = \begin{pmatrix} \xi^{11} & 0 & \cdots \\ \vdots & 0 & \cdots \\ \xi^{J1} & 0 & \cdots \end{pmatrix}_{J\times K}, \ \boldsymbol{\xi}^2 = \begin{pmatrix} 0 & \xi^{12} & \cdots \\ 0 & \vdots & \cdots \\ 0 & \xi^{J2} & \cdots \end{pmatrix}_{J\times K}, \ldots .$$

We collect factor demands in a $JK \times 1$ vector $\boldsymbol{n}^d$, and aggregate factor demand, aggregate factor supply, and factor endowments in $F \times 1$ vectors, $\boldsymbol{n}^{f,d}$, $\boldsymbol{n}^{f,s}$, and $\bar{\boldsymbol{n}}^{f,s}$, as follows:

$$\boldsymbol{n}^d = \begin{pmatrix} n^{11,d} \\ n^{21,d} \\ \vdots \\ n^{JF,d} \end{pmatrix}_{JF\times 1}, \quad \boldsymbol{n}^{f,d} = \begin{pmatrix} n^{1,d} \\ n^{2,d} \\ \vdots \\ n^{F,d} \end{pmatrix}_{F\times 1}, \quad \boldsymbol{n}^{f,s} = \begin{pmatrix} n^{1,s} \\ n^{2,s} \\ \vdots \\ n^{F,s} \end{pmatrix}_{F\times 1}, \quad \text{and} \quad \bar{\boldsymbol{n}}^{f,s} = \begin{pmatrix} \bar{n}^{1,s} \\ \bar{n}^{2,s} \\ \vdots \\ \bar{n}^{F,s} \end{pmatrix}_{F\times 1}.$$

We denote the element-wise derivatives of $\boldsymbol{\xi}$, $\boldsymbol{n}^d$, $\boldsymbol{n}^{f,d}$, $\boldsymbol{n}^{f,s}$, and $\bar{\boldsymbol{n}}^{f,s}$ by $\frac{d\boldsymbol{\xi}}{d\theta}$, $\frac{d\boldsymbol{n}^d}{d\theta}$, $\frac{d\boldsymbol{n}^{f,d}}{d\theta}$, $\frac{d\boldsymbol{n}^{f,s}}{d\theta}$, and $\frac{d\bar{\boldsymbol{n}}^{f,s}}{d\theta}$, respectively.

Marginal products/technology change. We collect marginal products of intermediates in a $J \times JK$ matrix $\boldsymbol{G}_x$ as follows:

$$\boldsymbol{G}_x = \begin{pmatrix} \boldsymbol{G}_x^1 & \cdots & \boldsymbol{G}_x^K \end{pmatrix}_{J\times JK} \quad \text{where} \quad \boldsymbol{G}_x^k = \begin{pmatrix} \frac{\partial G^1}{\partial x^{1k}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{\partial G^J}{\partial x^{Jk}} \end{pmatrix}_{J\times K}, \tag{35}$$

where $\boldsymbol{G}_x^k$ is a $J \times K$ matrix. Similarly, we collect marginal products of factors in a $J \times JF$ matrix $\boldsymbol{G}_n$ as follows:

$$\boldsymbol{G}_n = \begin{pmatrix} \boldsymbol{G}_n^1 & \cdots & \boldsymbol{G}_n^F \end{pmatrix}_{J\times JF} \quad \text{where} \quad \boldsymbol{G}_n^f = \begin{pmatrix} \frac{\partial G^1}{\partial n^{1f,d}} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \frac{\partial G^J}{\partial n^{Jf,d}} \end{pmatrix}_{J\times J}, \tag{36}$$

where $\boldsymbol{G}_n^f$ is a $J \times J$ matrix. We also collect the impact of technology changes in output in a $J \times 1$ vector $\boldsymbol{G}_\theta$, as follows:

$$\boldsymbol{G}_\theta = \begin{pmatrix} \frac{\partial G^1}{\partial \theta} \\ \vdots \\ \frac{\partial G^J}{\partial \theta} \end{pmatrix}_{J\times 1}.$$

Marginal rates of substitution. We collect conditional cross-sectional averages of marginal rates of substitution, defined in Equation (8), in $J \times 1$ and $F \times 1$ vectors $\boldsymbol{MRS}_c$ and $\boldsymbol{MRS}_n$, as follows:

$$\boldsymbol{MRS}_c = \begin{pmatrix} \mathbb{E}_i^{\pm}\left[MRS_c^{i,1}\right] \\ \vdots \\ \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] \end{pmatrix}_{J\times 1} \quad \text{and} \quad \boldsymbol{MRS}_n = \begin{pmatrix} \mathbb{E}_i^{\pm}\left[MRS_n^{i,1}\right] \\ \vdots \\ \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right] \end{pmatrix}_{F\times 1},$$

where

$$\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] = \mathbb{E}_{i|\frac{dc^{ij}}{d\theta}\neq 0}\left[MRS_c^{ij}\right] \quad \text{and} \quad \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right] = \mathbb{E}_{i|\frac{dn^{if}}{d\theta}\neq 0}\left[MRS_n^{if}\right].$$

Social net valuations. We can collect social net valuations in $JK \times 1$ and $JF \times 1$ vectors as follows:

$$\boldsymbol{\mu}_x = \boldsymbol{G}_x^T \boldsymbol{MRS}_c - \boldsymbol{1}_x \boldsymbol{MRS}_c \tag{37}$$

$$\boldsymbol{\mu}_n = \boldsymbol{G}_n^T \boldsymbol{MRS}_x - \boldsymbol{1}_n \boldsymbol{MRS}_n, \tag{38}$$

where $\boldsymbol{\mu}_x$ and $\boldsymbol{\mu}_n$ respectively denote stacked vectors of μ_x^{jk} and μ_n^{jf}, and $\boldsymbol{1}_x$ and $\boldsymbol{1}_n$ respectively denote $JK \times J$ and $JF \times F$ stacked matrices with appropriately defined columns of ones, described in the Appendix. The matrices $\boldsymbol{G}_x^T$ and $\boldsymbol{G}_n^T$ are transposes of $\boldsymbol{G}_x$ and $\boldsymbol{G}_n$, defined in

Equations (35) and (36). The matrices $\mathbf{1}_x$ and $\mathbf{1}_n$ are respectively given by

$$\mathbf{1}_x = \begin{pmatrix} \boldsymbol{u}_x^1 \\ \vdots \\ \boldsymbol{u}_x^K \end{pmatrix}_{JK\times K} \quad \text{where} \quad \boldsymbol{u}_x^1 = \begin{pmatrix} 1 & 0 & \cdots \\ \vdots & 0 & \cdots \\ 1 & 0 & \cdots \end{pmatrix}_{J\times K}, \ \boldsymbol{u}_x^2 = \begin{pmatrix} 0 & 1 & \cdots \\ 0 & \vdots & \cdots \\ 0 & 1 & \cdots \end{pmatrix}_{J\times K}, \ \ldots$$

$$\mathbf{1}_n = \begin{pmatrix} \boldsymbol{u}_n^1 \\ \vdots \\ \boldsymbol{u}_n^F \end{pmatrix}_{JF\times F} \quad \text{where} \quad \boldsymbol{u}_n^1 = \begin{pmatrix} 1 & 0 & \cdots \\ \vdots & 0 & \cdots \\ 1 & 0 & \cdots \end{pmatrix}_{J\times F}, \ \boldsymbol{u}_n^2 = \begin{pmatrix} 0 & 1 & \cdots \\ 0 & \vdots & \cdots \\ 0 & 1 & \cdots \end{pmatrix}_{J\times F}, \ \ldots .$$

B Proofs and Derivations: Sections 3 through 5

Proof of Lemma 1. (Welfare decomposition: aggregate efficiency vs. redistribution)

Proof. For any welfarist planner with Social Welfare Function $\mathcal{W}(V_1, \ldots, V_I)$, we can express $\frac{dW}{d\theta}$ as

$$\frac{dW}{d\theta} = \sum_i \frac{\partial \mathcal{W}}{\partial V_i}\frac{dV_i}{d\theta} = \sum_i \alpha^i \lambda^i \frac{\frac{dV_i}{d\theta}}{\lambda_i},$$

where $\alpha^i = \frac{\partial \mathcal{W}}{\partial V_i}$ and where λ^i is an individual normalizing factor that allows us to express individual welfare assessments into a common unit/numeraire. We can therefore write

$$\frac{dW^\lambda}{d\theta} = \frac{\frac{dW}{d\theta}}{\frac{\sum_i \alpha^i \lambda^i}{I}} = \sum_i \tilde{\omega}^i \frac{\frac{dV_i}{d\theta}}{\lambda_i} = \underbrace{\frac{\sum_i \tilde{\omega}^i}{I}}_{=1} \sum_i \frac{\frac{dV_i}{d\theta}}{\lambda_i} + I\mathbb{C}ov_i\left[\tilde{\omega}^i, \frac{\frac{dV_i}{d\theta}}{\lambda_i}\right] = \underbrace{\sum_i \frac{\frac{dV_i}{d\theta}}{\lambda_i}}_{=\Xi^{AE}} + \underbrace{\mathbb{C}ov_i^{\Sigma}\left[\tilde{\omega}^i, \frac{\frac{dV_i}{d\theta}}{\lambda_i}\right]}_{=\Xi_{RD}},$$

where $\tilde{\omega}^i = \frac{\alpha^i \lambda^i}{\frac{\sum_i \alpha^i \lambda^i}{I}}$, which implies that $\mathbb{E}_i\left[\tilde{\omega}^i\right] = \frac{\sum_i \tilde{\omega}^i}{I} = 1$. □

Proof of Lemma 2. (Aggregate efficiency decomposition: cross-sectional individual efficiency vs. production efficiency)

Proof. It follows from the proof of Proposition 1. □

Proof of Proposition 1. (Cross-sectional individual efficiency)

Proof. Given the definition of V_i in Equation (1), we can express $\frac{\frac{dV_i}{d\theta}}{\lambda_i}$ as

$$\frac{\frac{dV_i}{d\theta}}{\lambda_i} = \sum_j \frac{\frac{\partial u_i}{\partial c^{ij}}}{\lambda_i}\frac{dc^{ij}}{d\theta} + \sum_f \frac{\frac{\partial u_i}{\partial n^{if}}}{\lambda_i}\frac{dn^{if,s}}{d\theta} = \sum_j MRS_c^{ij}\frac{dc^{ij}}{d\theta} - \sum_f MRS_n^{if}\frac{dn^{if,s}}{d\theta},$$

where MRS_c^{ij} and MRS_n^{if} are defined in Equation (7). Hence, from Lemma 2, it follows that

$$\Xi^{AE} = \sum_i \frac{\frac{dV_i}{d\theta}}{\lambda_i} = \sum_j \sum_i MRS_c^{ij} \frac{dc^{ij}}{d\theta} - \sum_f \sum_i MRS_n^{if} \frac{dn^{if,s}}{d\theta}$$

$$= \underbrace{\sum_j \mathbb{E}_{i|\frac{dc^{ij}}{d\theta}\neq 0}\left[MRS_c^{ij}\right] \sum_i \frac{dc^{ij}}{d\theta} - \sum_f \mathbb{E}_{i|\frac{dn^{if,s}}{d\theta}\neq 0}\left[MRS_n^{if}\right] \sum_i \frac{dn^{if,s}}{d\theta}}_{=\Xi^{AE,P}}$$

$$\underbrace{+\sum_j \mathbb{C}ov^{\Sigma}_{i|\frac{dc^{ij}}{d\theta}\neq 0}\left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right] - \sum_f \mathbb{C}ov^{\Sigma}_{i|\frac{dn^{if,s}}{d\theta}\neq 0}\left[MRS_n^{if}, \frac{dn^{if,s}}{d\theta}\right]}_{=\Xi^{AE,I}}.$$

To simplify the notation, we introduce the operators $\mathbb{E}_i^{\pm}[\cdot]$ and $\mathbb{C}ov_i^{\Sigma\pm}[\cdot]$ as follows:

$$\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] = \mathbb{E}_{i|\frac{dc^{ij}}{d\theta}\neq 0}\left[MRS_c^{ij}\right] \quad \text{and} \quad \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right] = \mathbb{C}ov^{\Sigma}_{i|\frac{dc^{ij}}{d\theta}\neq 0}\left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right]$$

$$\mathbb{E}_i^{\pm}\left[MRS_n^{if}\right] = \mathbb{E}_{i|\frac{dn^{if,s}}{d\theta}\neq 0}\left[MRS_n^{if}\right] \quad \text{and} \quad \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_n^{if}, \frac{dn^{if,s}}{d\theta}\right] = \mathbb{C}ov^{\Sigma}_{i|\frac{dn^{if,s}}{d\theta}\neq 0}\left[MRS_n^{if}, \frac{dn^{if,s}}{d\theta}\right].$$

Hence, cross-sectional individual efficiency, $\Xi^{AE,I}$, can be expressed as[23]

$$\Xi^{AE,I} = \underbrace{\sum_j \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right]}_{\substack{\text{Cross-Sectional} \\ \text{Consumption Efficiency}}} - \underbrace{\sum_f \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_n^{if}, \frac{dn^{if,s}}{d\theta}\right]}_{\substack{\text{Cross-Sectional} \\ \text{Factor Supply Efficiency}}},$$

while production efficiency corresponds to

$$\Xi^{AE,P} = \underbrace{\sum_j \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] \frac{dc^j}{d\theta}}_{\substack{\text{Aggregate} \\ \text{Consumption Efficiency}}} - \underbrace{\sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right] \frac{dn^{f,s}}{d\theta}}_{\substack{\text{Aggregate} \\ \text{Factor Supply Efficiency}}}.$$

□

Proof of Proposition 2. (Properties of cross-sectional individual efficiency)

[23] By defining consumption shares $c^{ij} = \nu^{ij} c^j$, one could decompose cross-sectional individual consumption efficiency into reallocation and aggregate components, as follows:

$$\mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right] = \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{ij}, \frac{d\nu^{ij}}{d\theta}\right] c^j + \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{ij}, \nu^{ij}\right] \frac{dc^j}{d\theta},$$

where we use the fact that $\frac{dc^{ij}}{d\theta} = \frac{d\nu^{ij}}{d\theta} c^j + \nu^{ij} \frac{dc^j}{d\theta}$. A similar subdecomposition of cross-sectional factor supply efficiency exists. While these decompositions may be useful to further understand the sources of gains or losses for a particular perturbation, they are not particularly helpful to characterize efficiency conditions.

Proof. a) When $I = 1$, $\mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right] = \mathbb{C}ov_i^{\Sigma\pm}\left[MRS_n^{if}, \frac{dn^{if}}{d\theta}\right] = 0$ for all j and f.

b) When MRS_c^{ij} is identical for all i, $\mathbb{C}ov_i^{\Sigma\pm}\left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right] = 0$.

c) When MRS_n^{if} is identical for all f, $\mathbb{C}ov_i^{\Sigma\pm}\left[MRS_n^{if}, \frac{dn^{if}}{d\theta}\right] = 0$. □

Proof of Proposition 2. (Production Efficiency Decomposition #1: Value Added)

Proof. It follows from the proof of Proposition 1. □

Proof of Proposition 3. (Production Efficiency Decomposition #2: Levels of Intermediates)

Proof. In order to decompose production efficiency, $\Xi^{AE,P}$, we use the fact that the resource constraints for good j and factor f imply that

$$\frac{dc^j}{d\theta} = \sum_i \frac{dc^{ij}}{d\theta} = \frac{dy^j}{d\theta} - \sum_k \frac{dx^{kj}}{d\theta} \quad \text{and} \quad \frac{dn^{f,s}}{d\theta} = \sum_i \frac{dn^{if,s}}{d\theta} = \sum_j \frac{dn^{if,d}}{d\theta} - \frac{d\bar{n}^{f,s}(\theta)}{d\theta}.$$

The production function for good j implies that

$$\frac{dy^j}{d\theta} = \sum_k \frac{\partial G_j}{\partial x^{jk}} \frac{dx^{jk}}{d\theta} + \sum_f \frac{\partial G_j}{\partial n^{jf,d}} \frac{dn^{jf,d}}{d\theta} + \frac{\partial G^j}{\partial \theta}.$$

Hence, we can express $\frac{dc^j}{d\theta} = \sum_i \frac{dc^{ij}}{d\theta}$ as

$$\frac{dc^j}{d\theta} = \sum_k \left(\frac{\partial G_j}{\partial x^{jk}} \frac{dx^{jk}}{d\theta} - \frac{dx^{kj}}{d\theta} \right) + \sum_f \frac{\partial G_j}{\partial n^{jf,d}} \frac{dn^{jf,d}}{d\theta} + \frac{\partial G^j}{\partial \theta}.$$

First, we focus on the aggregate consumption efficiency term, which can be written as

$$\begin{aligned}
\sum_j \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] \frac{dc^j}{d\theta} &= \sum_j \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] \left(\sum_k \left(\frac{\partial G_j}{\partial x^{jk}} \frac{dx^{jk}}{d\theta} - \frac{dx^{kj}}{d\theta} \right) + \sum_f \frac{\partial G_j}{\partial n^{jf,d}} \frac{dn^{jf,d}}{d\theta} + \frac{\partial G^j}{\partial \theta} \right) \\
&= \sum_j \sum_k \left(\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] \frac{\partial G_j}{\partial x^{jk}} - \mathbb{E}_i^{\pm}\left[MRS_c^{i,k}\right] \right) \frac{dx^{jk}}{d\theta} \\
&+ \sum_j \sum_f \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] \frac{\partial G_j}{\partial n^{jf,d}} \frac{dn^{jf,d}}{d\theta} + \sum_j \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right] \frac{\partial G^j}{\partial \theta}.
\end{aligned}$$

Where the final equality uses the fact that

$$\sum_j \sum_k \underbrace{\mathbb{E}_{i|\frac{dc^{ij}}{d\theta} \neq 0}\left[MRS_c^{ij}\right]}_{=\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]} \frac{dx^{kj}}{d\theta} = \sum_j \sum_k \underbrace{\mathbb{E}_{i|\frac{dc^{ik}}{d\theta} \neq 0}\left[MRS_c^{i,k}\right]}_{=\mathbb{E}_i^{\pm}\left[MRS_c^{i,k}\right]} \frac{dx^{jk}}{d\theta}.$$

Next, we focus on the aggregate factor supply efficiency term, which can be written as

$$\sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\frac{dn^{f,s}}{d\theta} = \sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\left(\sum_j \frac{dn^{if,d}}{d\theta} - \frac{d\bar{n}^{f,s}(\theta)}{d\theta}\right)$$
$$= \sum_j\sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\frac{dn^{if,d}}{d\theta} - \sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\frac{d\bar{n}^{f,s}(\theta)}{d\theta}.$$

Combining these results, we can express $\Xi^{AE,P}$ as

$$\begin{aligned}\Xi^{AE,P} &= \sum_j \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{dc^j}{d\theta} - \sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\frac{dn^{f,s}}{d\theta}\\
&= \sum_j\sum_k \underbrace{\left(\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{\partial G_j}{\partial x^{jk}} - \mathbb{E}_i^{\pm}\left[MRS_c^{i,k}\right]\right)}_{=\mu_x^{jk}}\frac{dx^{jk}}{d\theta}\\
&+ \sum_j\sum_f \underbrace{\left(\mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{\partial G_j}{\partial n^{jf,d}} - \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\right)}_{=\mu_n^{jf}}\frac{dn^{jf,d}}{d\theta}\\
&+ \sum_j \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{\partial G^j}{\partial\theta} + \sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\frac{d\bar{n}^{f,s}(\theta)}{d\theta}.\end{aligned}$$

Hence, $\Xi^{AE,P}$ can be written as

$$\Xi^{AE,P} = \sum_j\sum_k \mu_x^{jk}\frac{dx^{jk}}{d\theta} + \sum_j\sum_f \mu_n^{jf}\frac{dn^{jf,d}}{d\theta} + \sum_j \mathbb{E}_i^{\pm}\left[MRS_c^{ij}\right]\frac{\partial G^j}{\partial\theta} + \sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\frac{d\bar{n}^{f,s}(\theta)}{d\theta},$$

where μ_x^{jk} and μ_n^{jf} are defined in Equations (11) and (12). This equation can be expressed in matrix form as

$$\Xi^{AE,P} = \boldsymbol{\mu}_x'\frac{d\boldsymbol{x}}{d\theta} + \boldsymbol{\mu}_n'\frac{d\boldsymbol{n}^d}{d\theta} + \boldsymbol{MRS}_x^T\boldsymbol{G}_\theta + \boldsymbol{MRS}_n'\frac{d\bar{\boldsymbol{n}}^{f,s}}{d\theta}, \tag{39}$$

where we can write

$$\boldsymbol{\mu}_x' = \boldsymbol{MRS}_c'\boldsymbol{G}_x - \boldsymbol{MRS}_c'\boldsymbol{1}_x^T$$
$$\boldsymbol{\mu}_n' = \boldsymbol{MRS}_x^T\boldsymbol{G}_n - \boldsymbol{MRS}_n'\boldsymbol{1}_n^T.$$

□

Proof of Proposition 5. (Intermediate Inverse Matrix)

Proof. From the definition of $x^{jk} = \xi^{jk}y^k$, it follows that

$$\frac{dx^{jk}}{d\theta} = \xi^{jk}\frac{dy^k}{d\theta} + \frac{d\xi^{jk}}{d\theta}y^k,$$

where $\frac{dy^j}{d\theta}$ is given by

$$\frac{dy^j}{d\theta} = \sum_k \frac{\partial G_j}{\partial x^{jk}} \frac{dx^{jk}}{d\theta} + \sum_f \frac{\partial G_j}{\partial n^{jf,d}} \frac{dn^{jf,d}}{d\theta} + \frac{\partial G^j}{\partial \theta}.$$

Combining both equations, we can write

$$\frac{dx^{jk}}{d\theta} = \underbrace{\xi^{jk} \sum_j \frac{\partial G^k}{\partial x^{kj}} \frac{dx^{kj}}{d\theta}}_{\text{Propagation}} + \underbrace{\frac{d\xi^{jk}}{d\theta} y^k + \xi^{jk} \left(\sum_f \frac{\partial G^k}{\partial n^{kf,d}} \frac{dn^{kf,d}}{d\theta} + \frac{\partial G^k}{\partial \theta} \right)}_{\text{Impulse}},$$

which can be equivalently expressed in vector/matrix form as follows:

$$\frac{d\boldsymbol{x}}{d\theta} = \underbrace{(\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x)^{-1}}_{\text{Propagation}} \underbrace{\left(\frac{d\boldsymbol{\xi}}{d\theta} \boldsymbol{y} + \boldsymbol{\xi} \left(\boldsymbol{G}_n \frac{d\boldsymbol{n}^d}{d\theta} + \boldsymbol{G}_\theta \right) \right)}_{\text{Impulse}}, \tag{40}$$

where all matrices are defined in Section A ☐

Proof of Proposition 6. (Production Efficiency Decomposition #3: Shares)

Proof. Combining Equations (39) and (40), we can express $\Xi^{AE,P}$ as

$$\begin{aligned}
\Xi^{AE,P} &= \boldsymbol{\mu}'_x \frac{d\boldsymbol{x}}{d\theta} + \boldsymbol{\mu}'_n \frac{d\boldsymbol{n}^d}{d\theta} + \boldsymbol{MRS}_x^T \boldsymbol{G}_\theta + \boldsymbol{MRS}'_n \frac{d\bar{\boldsymbol{n}}^{f,s}}{d\theta} \\
&= \boldsymbol{\mu}'_x (\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x)^{-1} \left(\frac{d\boldsymbol{\xi}}{d\theta} \boldsymbol{y} + \boldsymbol{\xi} \left(\boldsymbol{G}_n \frac{d\boldsymbol{n}^d}{d\theta} + \boldsymbol{G}_\theta \right) \right) + \boldsymbol{\mu}'_n \frac{d\boldsymbol{n}^d}{d\theta} + \boldsymbol{MRS}_x^T \boldsymbol{G}_\theta + \boldsymbol{MRS}'_n \frac{d\bar{\boldsymbol{n}}^{f,s}}{d\theta} \\
&= \underbrace{\boldsymbol{\mu}'_x (\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x)^{-1}}_{=\boldsymbol{\psi}'_x} \frac{d\boldsymbol{\xi}}{d\theta} \boldsymbol{y} + \underbrace{\left(\boldsymbol{\mu}'_x (\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x)^{-1} \boldsymbol{\xi} \boldsymbol{G}_n + \boldsymbol{\mu}'_n \right)}_{=\boldsymbol{\psi}'_n} \frac{d\boldsymbol{n}^d}{d\theta} \\
&+ \underbrace{\left(\boldsymbol{\mu}'_x (\boldsymbol{I}_{JK} - \boldsymbol{\xi} \boldsymbol{G}_x)^{-1} \boldsymbol{\xi} + \boldsymbol{MRS}'_c \right)}_{=\boldsymbol{\psi}'_G} \boldsymbol{G}_\theta + \boldsymbol{MRS}'_n \frac{d\bar{\boldsymbol{n}}^{f,s}}{d\theta} \\
&= \boldsymbol{\psi}'_x \frac{d\boldsymbol{\xi}}{d\theta} \boldsymbol{y} + \boldsymbol{\psi}'_n \frac{d\boldsymbol{n}^d}{d\theta} + \boldsymbol{\psi}'_G \boldsymbol{G}_\theta + \boldsymbol{MRS}'_n \frac{d\bar{\boldsymbol{n}}^{f,s}}{d\theta},
\end{aligned}$$

where $\boldsymbol{\psi}'_x$, $\boldsymbol{\psi}'_x$, and $\boldsymbol{\psi}'_x$ are defined in Equation (20), (21), and (22). In sum form, we can write $\Xi^{AE,P}$ as

$$\Xi^{AE,P} = \sum_j \sum_k \psi_x^{jk} \frac{d\xi^{jk}}{d\theta} y^k + \sum_j \sum_f \psi_n^{jf} \frac{dn^{jf,d}}{d\theta} + \sum_j \psi_G^j \frac{\partial G^j}{\partial \theta} + \sum_f \mathbb{E}_i^{\pm} \left[MRS_n^{if} \right] \frac{d\bar{n}^{f,s}(\theta)}{d\theta}$$

Using the fact that $\frac{d\xi^{jk}}{d\theta} = \frac{d\chi_x^{jk}}{d\theta}\phi_x^k + \chi_x^{jk}\frac{d\phi_x^k}{d\theta}$, we can write $\boldsymbol{\psi}_x' \frac{d\boldsymbol{\xi}}{d\theta}\boldsymbol{y} = \sum_j \sum_k \psi_x^{jk}\frac{d\xi^{jk}}{d\theta} y^k$ as

$$\begin{aligned}
\sum_j \sum_k \psi_x^{jk} \frac{d\xi^{jk}}{d\theta} y^k &= \sum_j \sum_k \psi_x^{jk} \frac{d\chi_x^{jk}}{d\theta}\phi_x^k + \sum_j \sum_k \psi_x^{jk}\chi_x^{jk}\frac{d\phi_x^k}{d\theta} y^k \\
&= \sum_k \left(\sum_j \psi_x^{jk}\frac{d\chi_x^{jk}}{d\theta}\right) x^k + \sum_k \left(\sum_j \chi_x^{jk}\psi_x^{jk}\right)\frac{d\phi_x^k}{d\theta} y^k \\
&= \sum_k \mathbb{C}ov_j^{\Sigma}\left[\psi_x^{jk}, \frac{d\chi_x^{jk}}{d\theta}\right] x^k + \sum_k \overline{\psi}_x^k \frac{d\phi_x^k}{d\theta} y^k,
\end{aligned}$$

where we define $\overline{\psi}_x^k = \sum_j \chi_x^{jk}\psi_x^{jk}$.

Using the fact that $\frac{dn^{jf,d}}{d\theta} = \frac{d\chi_n^{jf}}{d\theta} n^{f,d} + \chi_n^{jf}\frac{dn^{f,d}}{d\theta}$, we can write $\boldsymbol{\psi}_n' \frac{d\boldsymbol{n}^d}{d\theta} = \sum_j \sum_f \psi_n^{jf}\frac{dn^{jf,d}}{d\theta}$ as

$$\begin{aligned}
\sum_j \sum_f \psi_n^{jf}\frac{dn^{jf,d}}{d\theta} &= \sum_j \sum_f \psi_n^{jf}\frac{d\chi_n^{jf}}{d\theta} n^{f,d} + \sum_j \sum_f \psi_n^{jf}\chi_n^{jf,d}\frac{dn^{f,d}}{d\theta} \\
&= \sum_f \left(\sum_j \psi_n^{jf}\frac{d\chi_n^{jf}}{d\theta}\right) n^{f,d} + \sum_f \left(\sum_j \chi_n^{jf,d}\psi_n^{jf}\right)\frac{dn^{f,d}}{d\theta} \\
&= \sum_f \mathbb{C}ov_j^{\Sigma}\left[\psi_n^{jf}, \frac{d\chi_n^{jf}}{d\theta}\right] n^{f,d} + \sum_f \overline{\psi}_n^f \frac{dn^{f,d}}{d\theta}
\end{aligned}$$

where we define $\overline{\psi}_n^f = \sum_j \chi_n^{jf,d}\psi_n^{jf}$.

Finally, note that

$$\sum_f \overline{\psi}_n^f \frac{dn^{f,d}}{d\theta} + \sum_f \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]\frac{d\bar{n}^{f,s}(\theta)}{d\theta} = \sum_f \overline{\psi}_n^f \frac{dn^{f,s}(\theta)}{d\theta} + \sum_f \psi_{\bar{n}}^f \frac{d\bar{n}^{f,s}(\theta)}{d\theta},$$

where we define $\psi_{\bar{n}}^f = \overline{\psi}_n^f + \mathbb{E}_i^{\pm}\left[MRS_n^{if}\right]$. □

C Additional Results

C.1 As-if problems

C.1.1 Cross-Sectional Individual Consumption Efficiency

Suppose that a planner seeks to maximize the value of assigning consumption, c^{ij}, among individuals, taking as given aggregate c^j. Formally, this as-if planner solves

$$\begin{aligned}
&\max_{c^{ij}} \sum_i MRS_c^{ij} c^{ij} \\
&\text{s.t} \sum_i c^{ij} = c^j.
\end{aligned}$$

The Lagrangian of this problem is given by $\mathcal{L} = \sum_i MRS_c^{ij} c^{ij} - \eta \left(\sum_i c^{ij} - c^j\right)$, and the change in the objective by a feasible perturbation of c^{ij} satisfies

$$d\mathcal{L}^{c^j} = \sum_i \left(MRS_c^{ij} - \eta\right) \frac{dc^{ij}}{d\theta} = \sum_i MRS_c^{ij} \frac{dc^{ij}}{d\theta} = \mathbb{C}ov_i^{\Sigma} \left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right],$$

where we use the fact that $\sum_i \frac{dc^{ij}}{d\theta} = 0$. Hence, $\mathbb{C}ov_i^{\Sigma} \left[MRS_c^{ij}, \frac{dc^{ij}}{d\theta}\right] = 0$ is the condition for efficiency allocating a fixed amount of consumption c^j across different individuals.

C.1.2 Cross-Sectional Individual Factor Supply Efficiency

This derivation follows the same steps as the previous one. Suppose that a planner seeks to minimize the value of assigning factor supplies, $n^{if,s}$, among individuals, taking as given the total required aggregate factor supply $n^{f,s}$. Formally, this as-if planner solves

$$\min_{n^{if,s}} \sum_i MRS_n^{if,s} n^{if,s}$$
$$\text{s.t} \sum_i n^{if,s} = n^{f,s}.$$

The Lagrangian of this problem is given by $\mathcal{L} = \sum_i MRS_n^{if,s} n^{if,s} + \eta \left(\sum_i n^{if,s} - n^{f,s}\right)$, and the change in the objective by a feasible perturbation of c^{ij} satisfies

$$d\mathcal{L}^{n^{f,s}} = \sum_i \left(MRS_n^{if,s} + \eta\right) \frac{dn^{if,s}}{d\theta} = \sum_i MRS_n^{if,s} \frac{dn^{if,s}}{d\theta} = \mathbb{C}ov_i^{\Sigma} \left[MRS_n^{if,s}, \frac{dn^{if,s}}{d\theta}\right],$$

where we use the fact that $\sum_i \frac{dn^{if,s}}{d\theta} = 0$. Hence, $\mathbb{C}ov_i^{\Sigma} \left[MRS_n^{if,s}, \frac{dn^{if,s}}{d\theta}\right] = 0$ is the condition for efficiency allocating a fixed amount of factor supply c^j across different individuals. In this case, welfare increases with $-\mathbb{C}ov_i^{\Sigma} \left[MRS_n^{if,s}, \frac{dn^{if,s}}{d\theta}\right]$, hence the negative sign in Equation (9).

D Useful Mathematical Results

Covariance Decompositions without zeros. Consider two variables x_i and y_i. We define $\mathbb{E}_i[x] = \frac{\sum_i y_i}{I}$ and $\mathbb{E}_i[y] = \frac{\sum_i y_i}{I}$. We repeatedly use the following facts:

$$\sum_i x_i y_i = \underbrace{\sum_i (x_i - \mathbb{E}_i[x])(y_i - \mathbb{E}_i[y])}_{=\mathbb{C}ov_i^{\Sigma}[x,y]} + \frac{1}{I} \sum_i x_i \sum_i y_i,$$

where $\mathbb{C}ov_i^{\Sigma}[x,y] = I\mathbb{C}ov[x,y] = \sum_i (x_i - \mathbb{E}_i[x])(y_i - \mathbb{E}_i[y])$ and

$$\frac{1}{I} \sum_i x_i \sum_i y_i = \mathbb{E}_i[x] \sum_i y_i = \sum_i x_i \mathbb{E}_i[y].$$

Equivalently,

$$\mathbb{E}_i[xy] = \frac{\sum_i x_i y_i}{I} = \frac{\sum_i (x_i - \mathbb{E}_i[x])(y_i - \mathbb{E}_i[y])}{I} + \frac{\sum_i x_i}{I}\frac{\sum_i y_i}{I} = \mathbb{C}ov[x,y] + \mathbb{E}_i[x]\,\mathbb{E}_i[y].$$

Covariance Decompositions with zeros. When we are working with random variables with non-zero averages, we also use the fact that

$$\sum_i x_i y_i = \underbrace{\sum_{i|x_i\neq 0} (x_i - \mathbb{E}_i[x])(y_i - \mathbb{E}_i[y])}_{=\mathbb{C}ov^{\Sigma}_{i|x_i\neq 0}[x,y]} + \frac{1}{\sum_i \mathbb{I}[x_i \neq 0]} \sum_{i|x_i\neq 0} x_i \sum_{i|x_i\neq 0} y_i,$$

where $\mathbb{C}ov^{\Sigma}_{i|x_i\neq 0}[x,y] = \sum_{i|x_i\neq 0}(x_i - \mathbb{E}_i[x])(y_i - \mathbb{E}_i[y])$ and

$$\frac{1}{\sum_i \mathbb{I}[x_i \neq 0]} \sum_{i|x_i\neq 0} x_i \sum_{i|x_i\neq 0} y_i = \mathbb{E}_{i|x_i\neq 0}[x] \sum_{i|x_i\neq 0} y_i = \sum_{i|x_i\neq 0} x_i \mathbb{E}_{i|x_i\neq 0}[y].$$

E Applications

General Minimal Economy

The value of $\frac{dx^{13}}{d\theta}$ can be expressed as

$$\frac{dx^{13}}{d\theta} = \frac{d\xi^{13}}{d\theta}y^3 + \xi^{13}\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \xi^{13}\frac{\partial G^3}{\partial x^{33}}\frac{1}{1-\xi^{33}\frac{\partial G^3}{\partial x^{33}}}\left(\frac{d\xi^{33}}{d\theta}y^3 + \xi^{33}\left(\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \frac{\partial G^3}{\partial \theta}\right)\right) + \xi^{13}\frac{\partial G^3}{\partial \theta}.$$

In case of frictionless competitive markets, production efficiency is given by

$$\begin{aligned}
\Xi^{AE,P} &= \overbrace{\mu^{12}\frac{dx^{12}}{d\theta}}^{=0} + \mu^{13}\frac{dx^{13}}{d\theta} + \mu^{32}\frac{dx^{32}}{d\theta} + \overbrace{\mu^{33}\frac{dx^{33}}{d\theta}}^{=0} \\
&= \mu^{13}\frac{dx^{13}}{d\theta} + \mu^{32}\frac{dx^{32}}{d\theta} \\
&= p^3\frac{dx^{13}}{d\theta} - p^2\frac{dx^{32}}{d\theta},
\end{aligned}$$

where

$$
\begin{aligned}
\frac{dx^{13}}{d\theta} &= \frac{d\xi^{13}}{d\theta}y^3 + \xi^{13}\frac{\partial G^3}{\partial\theta} + \xi^{13}\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \xi^{13}\frac{\partial G^3}{\partial x^{33}}\frac{1}{1-\xi^{33}\frac{\partial G^3}{\partial x^{33}}}\left(\frac{d\xi^{33}}{d\theta}y^3 + \xi^{33}\left(\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \frac{\partial G^3}{\partial\theta}\right)\right) \\
&= \frac{d\xi^{13}}{d\theta}y^3 + \xi^{13}\frac{\partial G^3}{\partial\theta} + \xi^{13}\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \xi^{13}\frac{\partial G^3}{\partial x^{33}}\frac{1}{1-\xi^{33}\frac{\partial G^3}{\partial x^{33}}}\frac{d\xi^{33}}{d\theta}y^3 \\
&+ \xi^{13}\frac{\partial G^3}{\partial x^{33}}\frac{1}{1-\xi^{33}\frac{\partial G^3}{\partial x^{33}}}\xi^{33}\left(\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \frac{\partial G^3}{\partial\theta}\right) \\
&= \frac{d\xi^{13}}{d\theta}y^3 + \xi^{13}\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \frac{\xi^{13}\frac{\partial G^3}{\partial x^{33}}}{1-\xi^{33}\frac{\partial G^3}{\partial x^{33}}}\frac{d\xi^{33}}{d\theta}y^3 + \frac{\xi^{13}\frac{\partial G^3}{\partial x^{33}}}{1-\xi^{33}\frac{\partial G^3}{\partial x^{33}}}\xi^{33}\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 \\
&+ \xi^{13}\frac{\partial G^3}{\partial\theta} + \xi^{13}\frac{\partial G^3}{\partial x^{33}}\frac{1}{1-\xi^{33}\frac{\partial G^3}{\partial x^{33}}}\xi^{33}\frac{\partial G^3}{\partial\theta},
\end{aligned}
$$

and where using the fact that $\xi^{13}+\xi^{33}=1$ and $\xi^{13}=1-\xi^{33}$, we find that

$$
\begin{aligned}
\frac{dx^{13}}{d\theta} &= \overbrace{\left(\frac{d\xi^{13}}{d\theta} + \frac{d\xi^{33}}{d\theta}\right)}^{=0}y^3 + \frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \frac{\partial G^3}{\partial\theta} \\
&= \frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \frac{\partial G^3}{\partial\theta}.
\end{aligned}
$$

Finally, using the properties of frictionless competitive economies:

$$
\begin{aligned}
\Xi^{AE,P} &= p^3\frac{dx^{13}}{d\theta} - p^2\frac{dx^{32}}{d\theta} \\
&= p^3\left(\frac{\partial G^3}{\partial x^{32}}\frac{d\xi^{32}}{d\theta}y^2 + \frac{\partial G^3}{\partial\theta}\right) - p^2\frac{d\xi^{32}}{d\theta}y^2 \\
&= p^3\left(\frac{p^2}{p^3}\frac{d\xi^{32}}{d\theta}y^2 + \frac{\partial G^3}{\partial\theta}\right) - p^2\frac{d\xi^{32}}{d\theta}y^2 \\
&= p^3\frac{\partial G^3}{\partial\theta} + p^2\frac{d\xi^{32}}{d\theta}y^2 - p^2\frac{d\xi^{32}}{d\theta}y^2 \\
&= p^3\frac{\partial G^3}{\partial\theta}.
\end{aligned}
$$