ECON 500a
General Equilibrium and Welfare Economics
Asset Pricing

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Outline: Dynamic Stochastic Economies

1. Dynamic Economics
2. Stochastic Economics
3. Asset Pricing
4. Efficiency and Welfare
5. Incomplete Markets
6. Production, Firms, Ownership

- Readings
  - MWG: Chapter 19
  - Duffie (2001); Cochrane (2005); Campbell (2017)

Roadmap

1. Implications of competitive equilibrium for asset prices
   - Competitive Equilibrium $\Rightarrow$ Linear Pricing
   - Competitive Equilibrium $\Rightarrow$ No Arbitrage
2. Arbitrage pricing: Fundamental Theorem of Asset Pricing
   - No Arbitrage $\iff$ Linear Pricing
3. Fundamental Asset Pricing Equation: several versions
   i) Stochastic Discount Factor
   ii) State Prices
   iii) Risk-Neutral Probabilities
   iv) Beta Representation
4. Extensions
   - Replication: Binomial/Black-Scholes Model
   - Heterogeneous Beliefs
   - Beliefs/Preferences Equivalence
5. Application: Consumption Based Asset Pricing
6. Application: CAPM (Capital Asset Pricing Model)

Competitive Equilibrium implies Linear Pricing

- Asset prices in a competitive equilibrium satisfy linear pricing, that is, it is possible to find state-prices $\mu(s) > 0$, such that

$$q_0^z = \sum_s \mu(s)\, d_1^z(s)$$

  - In matrix form, $\boldsymbol{q}_0 = \boldsymbol{\mu}\boldsymbol{D}$, where

$$\boldsymbol{\mu} = (\mu(1), \ldots, \mu(s), \ldots, \mu(S))$$

    is a vector of state-prices of dimension $1 \times S$
  - Notation: $\mu(s)$ rather than $\mu_0(s)$
- Existence of $\mu(s)$ follows from optimality conditions (Euler equations)
- **Remark #1**: Many valid state-prices $\Rightarrow$ One per individual
  - Unique state-prices only when markets are complete
- **Remark #2**: this result requires having no portfolio constraints
  - No short-selling constraints
  - No borrowing constraints
  - No collateral constraints

Competitive Equilibrium implies No-Arbitrage

- *Absence of arbitrage*: A system of asset prices $\boldsymbol{q}_0$ is arbitrage free if there is no self-financing portfolio with positive payoffs, that is, if there is no portfolio $\boldsymbol{a}_0$ such that $\boldsymbol{q}_0\boldsymbol{a}_0 \leq 0$ and $\boldsymbol{D}\boldsymbol{a}_0 \geq 0$ (at least one with strict inequality)

$$\boldsymbol{D}\boldsymbol{a}_0 = \begin{pmatrix} \sum_z d_1^z(1)\, a_0^z \\ \vdots \\ \sum_z d_1^z(s)\, a_0^z \\ \vdots \\ \sum_z d_1^z(S)\, a_0^z \end{pmatrix}_{S\times 1}$$

- "No self-financing portfolio has weakly positive payoffs in every state and a strictly positive payoffs in some state"
- Absence of arbitrage $\rightarrow$ very weak restriction
  - Free-lunches are not available in financial markets
  - Applies to both complete and incomplete markets
- <u>Asset prices in a competitive equilibrium satisfy absence of arbitrage</u>
  - Proof: If preferences are strongly monotone and there are arbitrage opportunities, then individual demands are unbounded and the optimized value of individual problem is ∞

Arbitrage Pricing vs. Equilibrium Pricing

- Absence of arbitrage is a property of prices $\boldsymbol{q}_0$ and payoffs $\boldsymbol{D}$
  - No need to specify preferences (beyond non-satiation), technologies, or equilibrium notion
  - Applies to both complete and incomplete markets
  - No-arbitrage only informs us about relative prices
- Summers (1985): finance is "ketchup economics", criticizing financial economists methodological focus on arbitrage, neglecting broader economic fundamentals

  "There are ketchup economists who have shown that two-quart bottles of ketchup invariably sell for twice as much as one-quart bottles, except for deviations traceable to transaction costs. They conclude that the ketchup market is perfectly efficient. They ignore the forces of supply and demand and other economic fundamentals."

- Not us!

Fundamental Theorem of Asset Pricing

No Arbitrage $\iff$ Linear Pricing

1. Absence of arbitrage $\Rightarrow$ Linear Pricing: if $\boldsymbol{q}_0$ is arbitrage free, then $\boldsymbol{q}_0 = \boldsymbol{\mu}\boldsymbol{D}$
   - Proof: Farkas' lemma $\Rightarrow$ Theorem of the alternative
     See e.g. Duffie (2001)
2. Linear Pricing (with $\boldsymbol{\mu} > 0$) $\Rightarrow$ Absence of arbitrage
   - Proof: Linear pricing implies that $\boldsymbol{q}_0 = \boldsymbol{\mu}\boldsymbol{D}$, so the pricing of a portfolio is $\boldsymbol{q}_0\boldsymbol{a}_0 = \boldsymbol{\mu}\boldsymbol{D}\boldsymbol{a}_0$
   - Therefore if $\boldsymbol{D}\boldsymbol{a}_0 > 0$, and state prices are strictly positive, $\boldsymbol{\mu} > 0$, then $\boldsymbol{q}_0\boldsymbol{a}_0 > 0$, showing that no arbitrage opportunity can exist

- Economic insight: if individuals can freely buy and sell (shorting may be necessary) portfolios of assets, then asset prices must be linear in payoffs
  - Nonlinear pricing gives incentives to combine, slice, and build portfolios of assets to make arbitrage profits
  - Example: popcorn at baseball field

Summary

- Competitive Equilibrium $\Rightarrow$ Linear Pricing
- Competitive Equilibrium $\Rightarrow$ Absence of Arbitrage
- FTAP: Linear Pricing $\iff$ Absence of Arbitrage
  - Complete markets: unique linear pricing rule
  - Incomplete markets: many linear pricing rules
- **Remark**: FTAP does not involve equilibrium

Alternative Asset Pricing Formulations

- Linear pricing yields the Fundamental Asset Pricing Equation
  - As an equilibrium, or not
- Four formulations
  1. Stochastic discount factor
  2. State prices
  3. Risk-neutral probabilities
  4. Beta representation

i) Stochastic Discount Factor

$$\boxed{q_0^z = \sum_s \pi(s)\, m(s)\, d_1^z(s)} = \mathbb{E}\left[m(s)\, d_1^z(s)\right]$$

- $m(s)$ measures how valuable a payoff is (how hungry agents are)
  - $m(s)$ is high, marginal utility is high, bad state
  - $m(s)$ is low, marginal utility is low, good state
- Risk-free asset $(d_1^z(s) = 1) \Rightarrow q^f = \mathbb{E}[m(s)] = \frac{1}{1+r^f}$ (special notation)

$$\begin{aligned} q_0^z &= \mathbb{E}[m(s)]\,\mathbb{E}[d_1^z(s)] + \mathbb{C}ov\left[m(s), d_1^z(s)\right] \\ &= \underbrace{\frac{\mathbb{E}[d_1^z(s)]}{1+r^f}}_{\text{Discount Expected Payoff}} + \underbrace{\mathbb{C}ov\left[m(s), d_1^z(s)\right]}_{\text{Compensation for Risk}} \end{aligned}$$

  - If $\mathbb{C}ov\left[m(s), d_1^z(s)\right] > 0$ asset is a hedge
  - If $\mathbb{C}ov\left[m(s), d_1^z(s)\right] < 0$ asset is risky

ii) State Prices

$$\boxed{q_0^z = \sum_s \mu(s)\, d_1^z(s)}$$

- State-price (price of A-D security): $\mu(s) = \pi(s)\, m(s)$

$$q_0^z = \sum_s \underbrace{\pi(s)\, m(s)}_{=\mu(s)} d_1^z(s) = \sum_s \mu(s)\, d_1^z(s)$$

  - Risk-free rate is $q_0^f = \sum_s \mu(s) = \frac{1}{1+r^f}$

iii) Risk-Neutral Probabilities

$$\boxed{q_0^z} = \sum_s \mu(s)\, d_1^z(s) = \sum_s \mu(s) \sum_s \overbrace{\frac{\mu(s)}{\sum_s \mu(s)}}^{=\pi^\star(s)} d_1^z(s)$$

$$= \frac{\sum_s \pi^\star(s)\, d_1^z(s)}{1+r^f} = \boxed{\frac{\mathbb{E}^\star\left[d_1^z(s)\right]}{1+r^f}}$$

- $\pi^\star(s) = \frac{\mu(s)}{\sum_s \mu(s)}$ are called risk-neutral probabilities
  - They add up to one
  - They are not physical probabilities
  - They can be generated though a change of measure (Radon-Nikodym derivative, see next slide)
- Why are risk-neutral probabilities useful?
  - Under risk-neutral probabilities, all assets have an expected return equal to the risk-free rate:

$$\frac{\mathbb{E}^\star\left[d_1^z(s)\right]}{q_0^z} = 1 + r^f, \ \forall z$$

iii) Risk-Neutral Probabilities

- What is the interpretation of $\frac{\pi^{\star}(s)}{\pi(s)}$?

$$\frac{\pi^{\star}(s)}{\pi(s)} = \frac{m(s)}{\mathbb{E}[m(s)]} \iff \pi^{\star}(s) = \frac{m(s)}{\mathbb{E}[m(s)]}\pi(s)$$

  - $\frac{m(s)}{\mathbb{E}[m(s)]}$ is the Radon-Nikodym derivative
- If $m(s)$ is constant, then $\frac{\pi^{\star}(s)}{\pi(s)} = 1$. Otherwise:
  - States with low $m(s)$ relative to average (good states), have lower $\pi^{\star}(s)$ relative to $\pi(s)$
  - States with high $m(s)$ relative to average (bad states), have higher $\pi^{\star}(s)$ relative to $\pi(s)$
- Bad states are perceived as more likely if we insist on pricing assets by discounting cash flows at the risk-free rate

iv) Beta Representation

$$1 = \mathbb{E}\left[m(s)\frac{d_1^z(s)}{q_0^z}\right] = \mathbb{E}[m(s)]\,\mathbb{E}[R^z(s)] + \mathbb{C}ov[m(s), R^z(s)]$$

- Define $R^z(s) = \frac{d_1^z(s)}{q_0^z}$ and $R^f = 1 + r^f = \frac{1}{\mathbb{E}[m(s)]}$

$$\boxed{\underbrace{\mathbb{E}[R^z(s)] - R^f}_{\text{Risk Premium}}} = -\frac{\mathbb{C}ov[m(s), R^z(s)]}{\mathbb{E}[m(s)]}$$

$$= \boxed{\underbrace{\left(-\frac{\mathbb{C}ov[m(s), R^z(s)]}{\mathbb{V}ar[m(s)]}\right)}_{=\beta^z \text{ (Quantity of risk)}}\underbrace{\left(\frac{\mathbb{V}ar[m(s)]}{\mathbb{E}[m(s)]}\right)}_{=\lambda \text{ (Price of risk)}}}$$

- λ is called "price of risk" (same for all assets)
- β^z is called "quantity of risk" (regression coefficient)
- **Remark**: $\mathbb{V}ar[R^z(s)]$ does not pin down $\mathbb{E}[R^z(s)] - R^f$ directly
  - Covariances matter $\mathbb{C}ov[m(s), R^z(s)]$, not variances (!)

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Replication: Binomial/Black-Scholes Model

- Two-date, two-state, two-asset economy: $T = 1$, $S = Z = 2$
  - We seek to price a third asset via *replication*
- Asset 1: stock with price q_0^1 and final prices (or payoffs)

$$q_1^1(1) = hq_0^1 \quad \text{and} \quad q_1^1(2) = \ell q_0^1$$

- Asset 2: risk-free rate asset with interest rate

$$1 + r^f = \frac{1}{q_0^2}$$

- Absence of arbitrage requires $h > 1 + r^f > \ell > 0$.
  - If $1 + r^f > h$, shorting the stock and buying bonds $\Rightarrow$ Arbitrage
  - If $1 + r^f < \ell$, borrowing to buy the stock $\Rightarrow$ Arbitrage
- Payoffs of third asset: $d^3(1)$ and $d^3(2)$
  - What is the price of this asset q_0^3?
- Are markets complete here?

Replication: Binomial/Black-Scholes Model

- Replicating portfolio: a^1 shares of the stock and a^2 (face value of the) amount saved

$$a^1 h q_0^1 + a^2 = d^3(1) \quad (\text{state } s = 1)$$
$$a^1 \ell q_0^1 + a^2 = d^3(2) \quad (\text{state } s = 2)$$

- Solution to this system:

$$a^1 = \frac{d^3(1) - d^3(2)}{hq_0^1 - \ell q_0^1} \quad \text{and} \quad a^2 = \frac{hd^3(2) - \ell d^3(1)}{h - \ell}$$

- No arbitrage pricing requires that the price of asset to be replicated, q_0^3, must equal the value of the replicating portfolio. Therefore

$$q_0^3 = q_0^1 a^1 + q_0^2 a^2 \Rightarrow \boxed{q_0^3 = \frac{1}{1+r^f}\left(\pi^\star(1)\, d^3(1) + \pi^\star(2)\, d^3(2)\right)},$$

  - $\pi^\star(1) = \frac{1+r^f-\ell}{h-\ell}$ and $\pi^\star(2) = 1 - \pi^\star(1) = \frac{h-(1+r^f)}{h-\ell}$ are risk-neutral probabilities
  - $\mu(1) = \frac{1}{1+r^f}\pi^\star(1)$ and $\mu(2) = \frac{1}{1+r^f}\pi^\star(2)$ are state prices

Replication: Binomial/Black-Scholes Model

$$q_0^3 = \frac{1}{1+r^f}\left(\pi^\star(1)\, d^3(1) + \pi^\star(2)\, d^3(2)\right)$$

- We have found asset price in terms of q_0^1, h, ℓ, and r^f and payoffs
  - This formula can price <u>any</u> third derivative asset
- **Remark**: no need to specify physical probabilities of the states, $\pi(1)$ and $\pi(2)$ (!!!)
- But we cannot separate $\pi(s)$ from $m(s) \Leftarrow \mu(s) = \pi(s)\, m(s)$

Replication: Binomial/Black-Scholes Model

- Say we consider a scenario in which $h = 1.2$, $\ell = 0.8$, $q_0^f = 20$, and $1 + r^f = 1.12$
- Third asset is call option with strike $X = 23$
  - Payoffs are $d^3(1) = 1$ and $d^3(2) = 0$
- Risk-neutral probabilities are

$$\pi^\star(1) = \frac{1.12 - 0.8}{1.2 - 0.8} = 0.8 \quad \text{and} \quad \pi^\star(2) = 1 - \pi^\star(1) = 0.2$$

- Option price is

$$q_0^3 = \frac{1}{1.12}\left[0.8 \cdot 1 + 0.2 \cdot 0\right] = 0.71$$

- The logic underlying Black-Scholes-Merton formula is identical to the replication argument presented here
  (Black and Scholes, 1973; Merton, 1973)
- The Black-Scholes formula is the continuous time limit of the multi-period version of the pricing equation derived here

Heterogeneous Beliefs

- Preferences are now:

$$V^i = \sum_t \left(\beta^i\right)^t \sum_{s^t} \pi_t^i\left(s^t\right) u^i\left(c_t^i\left(s^t\right)\right)$$

- Positive results unchanged
  - Agreement on set of states with $\pi^i(s) > 0$ (absolute continuity)
- Note that

$$\pi_t\left(s^t\right)\tilde{u}^i\left(c_t^i\left(s^t\right)\right) = \pi_t\left(s^t\right)\underbrace{\frac{\pi_t^i\left(s^t\right)}{\pi_t\left(s^t\right)}u^i\left(c_t^i\left(s^t\right)\right)}_{=\tilde{u}^i\left(c_t^i\left(s^t\right)\right)}$$

Beliefs/Preferences Equivalence

- State prices are

$$\underbrace{\mu(s)}_{\text{state price}} = \underbrace{\pi(s)}_{\text{beliefs}} \underbrace{m(s)}_{\text{preferences}}$$

- Asset prices can be equally explained by beliefs or preferences
  - Shiller vs. Fama $\Rightarrow$ See Cochrane (2005) or Campbell (2017)
  - Purely looking at prices (LHS) cannot settle the debate
- Connection to Sonnenschein-Mantel-Debreu
  - Both results highlight large explanatory power of general equilibrium
  - Excess demand theorem hinges on $I \gg 1$
  - Beliefs/preferences equivalence applies even when $I = 1$

Consumption Based Asset Pricing: $I = 1$, $J = 1$, $S \geq 1$, $T \geq 1$

- Single-good, single individual endowment economy: $I = J = 1$
  Lucas (1978)
  - $I = 1$: representative agent $\rightarrow$ drop i superscript
- Resource constraints automatically imply that

$$c_0 = \bar{y}_0 \quad \text{and} \quad c_1(s) = \bar{y}_1(s)$$

- Asset prices (for any asset z):

$$q_0^z = \beta \sum_s \pi(s) \left(\frac{\bar{y}_1(s)}{\bar{y}_0} \right)^{-\gamma} d_1^z(s)$$

  - Written as a function of (aggregate) endowments, which are primitives
- What is $\mu(s)$? And $m(s)$?
  - In an equilibrium model we can separate $\pi(s)$ from $m(s)$
- Lucas *tree* is the particular asset that pays the aggregate endowment $d_1^z(s) = \bar{y}_1(s)$

CAPM: $I > 1$, $J = 1$, $S \geq 1$, $T = 1$

- $I > 1$, consumption only at date 1
- Start from beta representation

$$\mathbb{E}\left[R^z(s)\right] - R^f = \underbrace{\left(-\frac{\mathbb{C}ov\left[m(s), R^z(s)\right]}{\mathbb{V}ar\left[m(s)\right]}\right)}_{\beta^z}\underbrace{\left(\frac{\mathbb{V}ar\left[m(s)\right]}{\mathbb{E}\left[m(s)\right]}\right)}_{\lambda}$$

- Assume that $m(s) = a - R^M(s)$, where $R^M(s)$ is the return of the *market* portfolio, a portfolio of all assets in the economy
  - See e.g. Cochrane (2005) or Ingersoll (1987) for microfoundations (quadratic or CARA preferences)
- Note that

$$\beta^z = -\frac{\mathbb{C}ov\left[m(s), R^z(s)\right]}{\mathbb{V}ar\left[m(s)\right]} = \frac{\mathbb{C}ov\left[R^M(s), R^z(s)\right]}{\mathbb{V}ar\left[R^M(s)\right]}$$

- Applying this expression for the market portfolio, $z = M$, it must be that $\beta^M = 1$:

$$\mathbb{E}\left[R^M(s)\right] - R^f = \frac{\mathbb{V}ar\left[m(s)\right]}{\mathbb{E}\left[m(s)\right]} \Rightarrow \lambda$$

CAPM: Intuition

- Combining both results, we can derive the SML (security market line) prediction of the CAPM model:

$$\boxed{\mathbb{E}\left[R^z(s)\right] - R^f = \beta^z\left(\mathbb{E}\left[R^M(s)\right] - R^f\right)}$$

- The CAPM is derived from investors optimality condition for holding each asset z
- Assets with high payoffs/returns in good states (states in which $m(s)$ is low $\iff$ the market return is high) have a high β^z, and should have a low price in equilibrium, or equivalently, a high expected return
  - These are risky assets, investors demand a high expected return to hold these assets in equilibrium
- Assets with high payoffs/returns in bad states (states in which $m(s)$ is high $\iff$ the market return is low) have a low β^z, should have a high price in equilibrium, or equivalently, a low expected return
  - These are hedges, investors demand a low expected return to hold these assets in equilibrium

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