Prudential Policy with Distorted Beliefs*

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Abstract

This paper studies financial and monetary policy in environments in which equity investors and creditors may have distorted beliefs. We characterize conditions under which it is optimal to tighten or relax leverage caps in response to arbitrary changes in beliefs. The optimal policy response to belief distortions depends on the type as well as the extent of exuberance, and it is not generally true that regulators should lean against the wind by tightening leverage caps in response to optimism. We show that increased optimism by investors is associated with relaxing the optimal leverage cap, while increased optimism by creditors, or jointly by both investors and creditors is associated with a tighter optimal leverage cap. In the presence of government bailouts, increased optimism by equity investors may call for a tighter optimal leverage cap too, depending on whether equity optimism is concentrated on upside or downside risk. Increased optimism by either equity investors or creditors is associated with higher incentives to raise interest rates, so monetary tightening can act as a useful substitute for financial regulation.

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1 Introduction

A large part of financial policy is motivated by the concern that financial market participants might take on excessive levels of risk during boom periods. The most common narrative is that financial market participants are aware of the risks they are taking, but decide to take them anyway because they do not bear the full, society-wide downside of their actions. This can be because investors enjoy implicit government support (Farhi and Tirole, 2012), or because they fail to internalize the full macroeconomic costs of financial crises (Lorenzoni, 2008; Dávila and Korinek, 2017; Korinek and Simsek, 2016; Farhi and Werning, 2016). This insight underpins most of the literature that analyzes and calibrates optimal capital requirements and other macro-prudential policies.

However, some recent evidence suggests that financial investors might not be aware of the risks they are taking. Cheng et al. (2014) demonstrate that Wall Street insiders, even when trading on their personal account, did not act as if they knew of the risk of the 2008 housing crash. In credit markets more widely, Lopez-Salido et al. (2017) show that indicators of bond-market sentiment predict subsequent increases in credit spreads. Greenwood and Hanson (2013) and Baron and Xiong (2017) show that indicators of credit booms can be used to predict significant *negative* returns on bank equity and corporate bonds. This evidence is consistent with models in which market participants hold distorted beliefs. Despite its empirical relevance, there is little rigorous normative analysis of prudential policies in these environments.

In this paper, we study financial and monetary policy in environments in which equity investors and creditors may have distorted beliefs. We consider a tractable model in which equilibrium leverage and investment are endogenously determined as a function of the beliefs of investors and creditors over future states of nature. A critical feature of our analysis is that we do not restrict a priori the shape of the distributions of investors' and creditors' beliefs. Our model can therefore be used flexibly to consider the consequences of any heuristic or bias. For example, investors and/or creditors may overstate the expected value of investment returns, understate their variance, or downplay the likelihood of rare shocks.

We initially study the impact of changes in beliefs on equilibrium leverage and investment.[1] Our general variational characterization highlights that the impact of optimism on equilibrium leverage and investment depends non-trivially on the exact form of optimism. For instance, creditors' belief distortions near the default boundary are particularly important when distress costs are large, and investors' distortions about downside (default) states

[1] Formally, we analyze a model with limited participation, in which investors cannot issue equity to creditors. Belief distortions in this environment break the Modigliani-Miller theorem, generating a meaningful leverage choice for investors. We allow for two further canonical frictions: Default is associated with deadweight losses, and investors may be relatively impatient.

are not relevant for market valuations.[2] In our model, equilibrium investment choices are determined by a levered version of Tobin's q, which measures the market value of the investors' securities per unit of investment. Equilibrium leverage choices are driven by the marginal change in market value when investors borrow more. The effects of belief distortions on this statistic and, consequently, on leverage are nuanced: There is a fundamental asymmetry whereby optimism (in a hazard rate sense) among creditors increases the marginal value of leverage, while optimism among equity investors decreases it. Moreover, perhaps surprisingly, when equity investors and creditors share the same distorted beliefs, the behavior of leverage is qualitatively the same as in the case in which only creditors have distorted beliefs.

We then present our normative results, which are the central contribution of this paper. We study the second-best problem of a planner who can impose a cap on the leverage ratio of investors, but cannot control the level of risky investment. The planner adopts a paternalistic approach and evaluates agents' utility according to probability distributions that may differ from their own. We initially study an environment in which belief distortions are the only reason for policy intervention, and then introduce government bailouts, which provide an additional rationale for intervention.

We show that the marginal benefit of permitting more leverage is the sum of two components. The first is the *inframarginal* effect of more leverage on existing units of investment. This term relates to the common intuition that tighter leverage caps provide a buffer against the social costs of distress. The second is the *incentive* effect, which arises because the level of permitted leverage changes investors' incentives to invest. For example, a tighter leverage cap disciplines investment when it forces exuberant investors to have more "skin in the game." The incentive effect, in turn, hinges on the sensitivity of the investors' investment to leverage policy.

Our central normative result determines the desirability of tightening or relaxing leverage caps in response to arbitrary distortions in beliefs. In response to belief distortions, the inframarginal effect implies tighter optimal leverage caps if *(i)* the distortion increases privately optimal investment (i.e., the number of inframarginal units) and *(ii)* the private sector overstates the marginal benefit of leverage under the planner's beliefs. The incentive effect encourages tighter leverage regulation if *(i)* belief distortions reduce the sensitivity of investment to leverage policy and *(ii)* Tobin's q is smaller (below 1) when evaluated using the planner's beliefs.

We first characterize these effects using general variational methods, which can be used to evaluate the consequences of arbitrary belief distortions for optimal policy. We then characterize optimal policy responses in three more concrete scenarios. In an *equity exuberance*

[2] The upside/downside distinction is related to the analysis of Simsek (2013a), but not identical. We explicitly relate our positive results to his in the Appendix.

scenario, creditors' beliefs agree with the planner's but equity investors are more optimistic in a hazard rate sense. In a *debt exuberance* scenario, investors agree with the planner and creditors are more optimistic in a hazard rate sense. In a *joint exuberance* scenario, both creditors and investors are more optimistic than the planner in a hazard rate sense.

Despite the subtlety of the optimal policy in general, our model gives sharp policy implications in these scenarios. First, optimism in an equity exuberance scenario removes all incentives to constrain leverage.[3] Intuitively, this is the result of two forces. First of all, the planner wishes to push investors towards issuing more debt and less equity against inframarginal units of investment, because optimistic equity investors (wrongly) consider debt to be undervalued. Moreover, equity optimism weakens the planner's desire to discipline investment by reducing the sensitivity of investment to leverage caps. By contrast, increased optimism in a debt exuberance scenario always leads to stricter optimal leverage caps, because it leads to an overvaluation of debt and increases the sensitivity of investment to leverage policy. Finally, as discussed above, debt dominates marginal valuations in the joint exuberance scenario, so that it also leads to stricter optimal regulation.

We consider three extensions to our baseline model. The first extension introduces the possibility that the government provides bailouts to investors ex-post. This additional friction leads to several new insights, which are particularly useful in the context of leverage policy in banking. First, we show that when bailouts are a convex and decreasing function of realized investment returns, belief distortions in good states of the world become especially important for policy. Second, we show that bailouts can reverse our baseline result for the equity exuberance scenario. Intuitively, the planner now has a stronger incentive to prevent increases in leverage on inframarginal units of investment, which would raise the deadweight fiscal costs of bailouts. In this context, the type of equity distortion becomes crucial, as we demonstrate in the classical case where investors are "too big to fail". If equity exuberance mainly overstates large upside returns in solvent states of the world, as opposed to neglecting downside risk, then the inframarginal effect dominates and it becomes optimal to impose stricter leverage caps.

By contrast, if equity exuberance focuses on downside risk, then the incentive effect dominates. Since leverage policy becomes a blunt tool for investment incentives when equity investors are optimistic, the optimal policy response in this case is to relax leverage caps. An additional, positive implication of this result is that strict capital requirements in the banking sector need not curb the most severe credit cycles, because the sensitivity of investment to leverage is muted in exuberant times. This goes some way towards reconciling the empirical evidence: Capital requirements are effective for incentives on average (e.g., Jiménez et al.,

[3] Indeed, this scenario generates a case for leverage floors or, conversely, limits on equity issuance. Our results can be used to rationalize recent policy interventions that limit equity issuance, as explained in Page 22.

2014), but not to smooth out the largest booms and busts (e.g., Jorda et al., 2017).

In the second extension, the government has the ability to affect the interest rates on investors' debt using monetary policy. Even in exuberant times, equity investors remain sensitive to monetary tightening (an increase in interest rates) because this policy raises the cost of leverage for a solvent firm. Crucially, the role of beliefs for the response to monetary policy is *opposite* from the response to leverage regulation. The leverage cost increase is especially important for investors who neglect the possibility of failure, because they expect the cost of leverage to come out of their own pocket, as opposed to the taxpayer's. Formally, we show that the marginal welfare benefit of monetary tightening can increase with exuberance, and does so precisely in situations where the benefit of capital regulation declines. These results connect our paper to the literature on monetary policy as a prudential tool. Monetary policy has been advocated in situations where traditional financial regulation cannot reach the "shadow banking" sector, or is otherwise constrained (e.g., Stein, 2013; Caballero and Simsek, 2019). Even in a model without such constraints, we show that monetary policy is useful because it reins in exuberant credit booms, and is particularly effective at times when capital regulation is endogenously constrained by distorted beliefs.

Our final extension relaxes the assumption of paternalism by considering imperfectly targeted policy. We introduce a random variable indexing sentiments among investors and creditors, and assume that the government must commit to a leverage cap before sentiments are realized. In this environment, the government is aware of potential belief disortions, but cannot detect or respond to them in real time. We compare the welfare effect of leverage regulation in this extension to a benchmark with perfect targeting. In particular, the *ex ante* effect of leverage regulation depends on the covariance (across realizations of sentiment) of the desirability and effectiveness of policy. For example, in the case of sentiments among equity investors, leverage caps are least desirable in optimistic states of the world, which is also when investors are least sensitive to the cap. This covariance reduces the planner's incentive to impose a cap. By constrast, in the case of sentiments among creditors, optimism coincides with high sensitivity and a strong incentive to regulate, so that the covariance is reversed and pushes for tighter leverage regulation.[4]

Our paper is related to several literatures. Our approach to computing welfare is related to growing literature that explores the normative implications of belief heterogeneity. Brunnermeier et al. (2014) develop a criterion to detect speculation under heterogeneous beliefs, which is also used in Simsek (2013b) and Caballero and Simsek (2020) to provide normative assessments of financial innovation and stabilization policy, respectively. Dávila (2014) characterizes the optimal financial transaction tax for a paternalistic planner in an

[4] In addition, the leverage cap may not always bind in the case with imperfect targeting. We show that this effect generally has ambiguous effects on optimal policy.

environment with heterogeneous beliefs. Campbell (2016), Farhi and Gabaix (2017) and Exler et al. (2019) also explore paternalistic policies in a household context, while Haddad et al. (2020) do so in the context of technological innovations.

The bulk of the work that studies the relationship between beliefs and leverage, including the contributions of Geanakoplos (1997), Fostel and Geanakoplos (2008, 2012, 2015, 2016), Simsek (2013a), and Bailey et al. (2019), has been carried out in models of collateralized credit. As we show in the Appendix, our results carry through unchanged to that case. Our results are connected to the by now well-developed literature on government bailouts, which includes the recent contributions of Farhi and Tirole (2012), Bianchi (2016), Chari and Kehoe (2016), Gourinchas and Martin (2017), Cordella et al. 2018, Dávila and Walther (2020) and Dovis and Kirpalani (2020), among others. We provide a novel analysis of how bailouts and belief distortions interact, and how they jointly shape the optimal regulatory policy.

Finally, our results also contribute to the literature that explores the interaction between monetary and regulatory policy. The recent work of Caballero and Simsek (2019) is the closest to ours. While they study the design of macroprudential and monetary policy in a model with nominal rigidities and aggregate demand effects, we instead think about optimal policies in a model of risky credit with a rich specification of beliefs.

The structure of the paper is as follows: Section 2 introduces our baseline model, characterizes the model equilibrium, and describes some key positive properties of the model. Section 3 presents the central welfare effects that determine the optimal leverage regulation. Section 4 extends our results to an en environment with government bailouts, while Section 5 considers the role of monetary policy. Section 7 concludes. All proofs and derivations are in the Appendix.

2 Baseline model

We initially study how beliefs impact financial regulatory policy abstracting from government bailouts and monetary policy. In Sections 4 and 5, we extend our model to incorporate both.

2.1 Environment

Agents, preferences and endowments. There are two dates, indexed by 0 and 1, and a single consumption good (dollar), which serves as numeraire. There are three types of agents: A unit measure of investors, indexed by I, a unit measure of creditors, indexed by C, and a government, which sets financial policy and monetary policy. We denote the possible states of nature at date 1 by s, which corresponds to the realization of the investors' technology, as described below. We assume that $s \in [\underline{s}, \overline{s}]$, where $\underline{s} > 0$.

Both investors and creditors are risk-neutral. The lifetime utility of investors is given by $c_0^I + \beta^I \mathbb{E}^I \left[c_1^I(s) \right]$, where c_0^I and $c_1^I(s)$ denote the consumption of investors and $\mathbb{E}^I[\cdot]$ denotes the expectation under the investors' beliefs, whose determination is described below. The lifetime utility of creditors is given by $c_0^C + \beta^C \mathbb{E}^C \left[c_1^C(s) \right]$, where c_0^C and $c_1^C(s)$ denote the consumption of creditors and $\mathbb{E}^C[\cdot]$ denotes the expectation under the creditors' beliefs. We assume that $0 < \beta^I \leq \beta^C \leq 1$, so that investors are more impatient than creditors. This assumption generates gains from trade: It is better for creditors to finance up front investments because they discount the future less.

The endowments of the consumption good of investors and creditors at dates 0 and 1 are respectively given by $\left\{ w_0^I, w_1^I(s) \right\}$ and $\left\{ w_0^C, w_1^C(s) \right\}$. Creditors' and investors' endowments are large enough so that their consumption never becomes negative.[5]

Investment technology. Investors can invest at date 0 to create $k \geq 0$ units of productive capital. This investment in capital yields sk dollars in state s at date 1. As in canonical "Tobin's q" models of investment, investment at date 0 costs $\Upsilon(k)$ dollars, where $\Upsilon(k)$ is a convex adjustment cost that satisfies $\Upsilon(0) = 0$, $\lim_{k \to 0} \Upsilon'(k) = 0$, $\Upsilon'(k) \geq 0$, and $\Upsilon''(k) \geq 0$.

Financial contracts. Investors finance their investment by issuing bonds with face value b per unit of investment (i.e., the total stock of debt issued is bk, and an investor's leverage ratio is simply b). Any remaining financing is obtained with an equity contribution from the investor's endowment. Because investors are more impatient than creditors, they perceive bond finance to be cheaper than equity. The difference in time preferences $\beta^C - \beta^I$ can be interpreted as a cost of equity issuance.[6] The key assumption on financing sustained throughout the paper is limited participation: creditors cannot fund investors using equity.

The difference in discount factors between investors and creditors guarantees that the investors' problem is well-behaved, as illustrated below, but our results also hold in environments in which belief differences are the single rationale for investors to borrow, under suitable regularity conditions. In the Appendix, we show that it is straightforward to add outside equityholders with the same discount factor and beliefs as investors.

At date 1, after the state s is realized, investors decide whether to default. If investors

[5] In the Appendix, we study an alternative scenario in which the non-negativity constraint of investors' consumption at date 0 binds. We show that this case, in which the equity contribution of investors is effectively capped, our positive analysis maps to the model in Simsek (2013a).

[6] There are readily available theories that can generate a cost of equity issuance or, equivalently, a benefit from issuing debt. For example, moral hazard among shareholders, a demand for "money-like" claims (Gorton and Pennacchi, 1990; Stein, 2012; DeAngelo and Stulz, 2015), or bank runs and market discipline (Diamond and Rajan, 2001).

default, creditors seize all of the investors' resources and receive ϕs per unit of investment, where $0 \leq \phi \leq 1$. The remainder $(1-\phi)s$ measures the deadweight loss or cost of distress associated with default.

Beliefs. We adopt a flexible approach to model the perceptions of investors and creditors over future states of nature. Formally, we assume that investors perceive the distribution over future states $s \in [\underline{s}, \overline{s}]$ to be $F^I(s)$, while creditors perceive it to be $F^C(s)$. The distributions $F^I(s)$ and $F^C(s)$ can differ from each other and from the true distribution, which we denote by $F(s)$. The advantage of this flexible approach is that it allows us to analyze the consequences of different biases studied in behavioral economics. This would conclude the description of the model if there were no government intervention.

Financial regulatory policy. The government is able to impose a leverage cap on investors at date 0. This cap is the central object of study in this paper. The government requires that investors set $b \leq \bar{b}$, where $1-\bar{b}$ is the minimal permitted ratio of equity contribution to risky investment. This constraint imposes a debt limit per unit of risky investment, or equivalently, a minimal equity contribution. If investors are interpreted as banks, then the constraint corresponds to a standard capital adequacy requirement with a positive risk weight on risky investment. We focus our attention on the case in which the government cannot directly control the scale of investors' risky investment k, which forces the government to face a second-best policy problem.[7] We discuss the form of the first-best policy in the Appendix E.2.

Equilibrium definition. Given a regulatory debt limit $\bar{b}$, an *equilibrium* in this economy is defined by an investment choice $k \geq 0$ and a leverage choice $b \leq \bar{b}$ that maximize the investors' expected utility taking into account that the debt issued by investors is priced competitively by creditors.

[7]It is possible to justify the assumption that the government cannot control the scale of investment. For example, "nationalization" policies that control every one of investors' decisions are not optimal when private agents have real-time information about investment opportunities that the government does not have (e.g., Walther, 2015). Perhaps for this reason, all relevant regulatory constraints in practice (e.g., capital requirements, leverage, liquidity coverage, and net stable funding requirements in Basel III) focus on ratios of bank assets to liabilities. Similarly, regulations on creditors focus on loan-to-value and debt-to-income ratios. All of these instruments leave the dollar amount of investors' investments as a free variable. As an alternative, one could consider a model where investors can engage in asset substitution (or "risk shifting"), which is typically modeled as a situation where investors can increase the riskiness of a portfolio of fixed scale, but where the regulator cannot observe this choice (e.g., Allen and Gale, 2000; Repullo, 2004). Similar insights emerge in this case.

2.2 Equilibrium characterization

We initially characterize the investors' default decision at date 1 and then the investors' borrowing and investment choices at date 0, which depend on the creditors' debt pricing decision.

The default decision of investors at date 1 takes a threshold form and can be characterized as follows

$$\begin{aligned} &\text{if } s < b, \quad \text{Default} \\ &\text{if } s \geq b, \quad \text{No Default.} \end{aligned} \tag{1}$$

Consequently, there is unique threshold $s^{\star}(b) = b$, such that investors default if $s < s^{\star}(b)$ and repay otherwise. We use the more general notation $s^{\star}(b)$ throughout — instead of simply using b as default threshold — to highlight that the default boundary $s^{\star}(b)$ is in general a non-linear function of b, as shown in our analysis of ex-post government interventions in Section 4.

We begin by characterizing the maximization problem of investors:

Lemma 1. *[Investors' problem] Investors solve the following problem to decide their optimal investment and leverage choices at date* 0*:*

$$V\left(\bar{b}\right) = \max_{b,k} \left[M(b) - 1\right] k - \Upsilon(k) \tag{2}$$

$$s.t. \quad b \leq \bar{b} \qquad\qquad (\mu), \tag{3}$$

where μ denotes the Lagrange multiplier on the leverage constraint imposed by the government (reformulated as $bk \leq \bar{b}k$), and $M(b)$ is given by

$$M(b) = \underbrace{\beta^I \int_{s^{\star}(b)}^{\bar{s}} (s-b)\, dF^I(s)}_{equity} + \underbrace{\beta^C \left(\int_{s^{\star}(b)}^{\bar{s}} b dF^C + \phi \int_{\underline{s}}^{s^{\star}(b)} s dF^C(s) \right)}_{debt}. \tag{4}$$

The value function $V\left(\bar{b}\right)$ in Equation (2) measures the net present-value of the investment as a function of the leverage constraint $\bar{b}$, while $M(b)$ corresponds to the market value of equity and debt (per unit of investment) after investing as a function of the leverage ratio b. The first term in Equation (4) corresponds to the present-value of the equity payoffs, as perceived by investors. The second term in Equation (4) corresponds to the present-value of the debt payoffs, as perceived by creditors. While debt payoffs are priced using the creditors' discount factor β^C and beliefs $F^C(s)$, equity payoffs are priced using the investors' discount factor β^I and beliefs $F^I(s)$.

Note that the market value of debt and equity in Equation (4) can be equivalently expressed as follows

$$M(b) = \underbrace{\beta^C \int_{\underline{s}}^{\overline{s}} s dF^C(s)}_{\text{MM valuation}} - \underbrace{\int_{s^\star(b)}^{\overline{s}} (s-b)\left(\beta^C dF^C(s) - \beta^I dF^I(s)\right)}_{\text{excess cost of equity}} - \underbrace{(1-\phi)\,\beta^C \int_{\underline{s}}^{s^\star(b)} s dF^C(s)}_{\text{cost of distress}}. \tag{5}$$

Equation (5) clearly illustrates the forces that determine the equilibrium choices of b and k, which we characterize in Proposition 1 below. The first term in Equation (5) corresponds to the valuation of all of the firm's cash flows from the perspective of creditors. This term is independent of leverage and investment decisions by the Modigliani-Miller theorem.[8] The second term in Equation (5) captures the differential valuation of cash flows in solvent states by creditors and investors. This term represents an excess cost of equity due to the relative impatience of investors, but belief differences can attenuate this cost, for example, when investors are more optimistic about cash flows than creditors. The last term in Equation (5) captures the net present-value of the cost of distress after default. Notice that we recover the Modigliani-Miller theorem when i) there is no cost of distress, $\phi = 1$, and ii) investors and creditors value cash flows equally, $\beta^I = \beta^C$ and $F^I(\cdot) = F^C(\cdot)$. In this special case, the choice of b is indeterminate.

We now characterize optimal choices by investors.

Proposition 1. *a) [Leverage Choice] Equilibrium leverage $b^\star$ is given by the solution to*

$$\frac{dM}{db}(b^\star) = \mu, \tag{6}$$

where

$$\frac{dM}{db}(b) = \underbrace{\beta^C \int_{s^\star(b)}^{\overline{s}} dF^C(s) - \beta^I \int_{s^\star(b)}^{\overline{s}} dF^I(s)}_{\textit{mg. cost of equity}} - \underbrace{(1-\phi)\,\beta^C s^\star(b)\, f^C(s^\star(b))}_{\substack{\textit{mg. cost of distress} \\ \textit{(boundary)}}}. \tag{7}$$

When the investors' leverage constraint doesn't bind, $b^\star < \bar{b}$ and $\mu = 0$. When the leverage constraint binds, $b^\star = \bar{b}$ and $\mu > 0$.

b) [Investment Choice] Regardless of whether the investors' leverage constraint binds or not, equilibrium investment $k^\star$ is given by the solution to

$$M(b^\star) - 1 = \Upsilon'(k^\star), \tag{8}$$

[8] We use the creditors' discount factor and beliefs as a reference to express the Modigliani-Miller valuation term. There is an equivalent formulation of Equation (5) that uses the discount factor and beliefs of the investors as reference.

where $b^\star$ satisfies Equations (6) and (7).

Two forces determine the private marginal value of leverage per unit of investment in Equation (7). The first force arises due to the differences in valuation between investors and creditors. By increasing the leverage ratio b, an investor is able to raise in present-value terms $\beta^C \int_{s^\star(b)}^{\overline{s}} dF^C(s)$ dollars, whose repayment cost in present-value terms for investors corresponds to $\beta^I \int_{s^\star(b)}^{\overline{s}} dF^I(s)$. The second force corresponds to the marginal reduction in the value of the firm caused by defaulting more frequently after increasing the leverage ratio. At an interior optimum, investors optimally trade off these forces by setting $b^\star$ so that $\frac{dM}{db}(b^\star) = 0$. When the leverage constraint binds, $b^\star = \bar{b}$, and Equation (7) simply defines the positive multiplier μ. Note that Equation (6) fully determines $b^\star$ separately from $k^\star$.

Equation (8), which is a levered version of Tobin's marginal q, characterizes the optimal investment decision. Its left-hand side, $M(b^\star) - 1$, measures the private value to equity-holders of owning an additional unit of productive capital for a given level of leverage. Its right-hand side, $\Upsilon'(k^\star)$, simply corresponds to the marginal cost of investment.

In the Appendix, we formalize the regularity conditions that guarantee that the optimal leverage choice $b^\star$ is positive and finite. There we show that a sufficient condition for an interior optimum for $b^\star$ and $k^\star$ to exist without leverage regulation is that $\beta^C \phi \mathbb{E}^C[s] < 1$. On the one hand, the difference in discount factors motivates investors to choose positive leverage, since

$$\left.\frac{dM}{db}(b)\right|_{b=0} = \beta^C - \beta^I > 0,$$

as implied by Equation (7). On the other hand, the condition that we identify guarantees that infinite borrowing is not optimal. Going forward, we proceed as if the laissez-faire equilibrium is reached at an interior maximum, in which $\frac{d^2M}{db^2}(b^\star) < 0$.

2.3 Comparative statics

We now characterize several properties of the equilibrium that will inform our normative results. We initially characterize the sensitivity of equilibrium investment to changes in the leverage constraint in the following Lemma.

Lemma 2. *[Sensitivity of investment to leverage constraint] The sensitivity of investors' investment to the leverage constraint is given by*

$$\frac{dk^\star}{d\bar{b}} = \frac{\mu}{\Upsilon''\left(k^\star\left(\bar{b}\right)\right)} \geq 0, \tag{9}$$

where μ satisfies Equation (6) and $k^\star\left(\bar{b}\right)$ denotes the optimal investment choice as a function of $\bar{b}$, characterized in Equation (8). Hence, relaxing (tightening) the leverage constraint

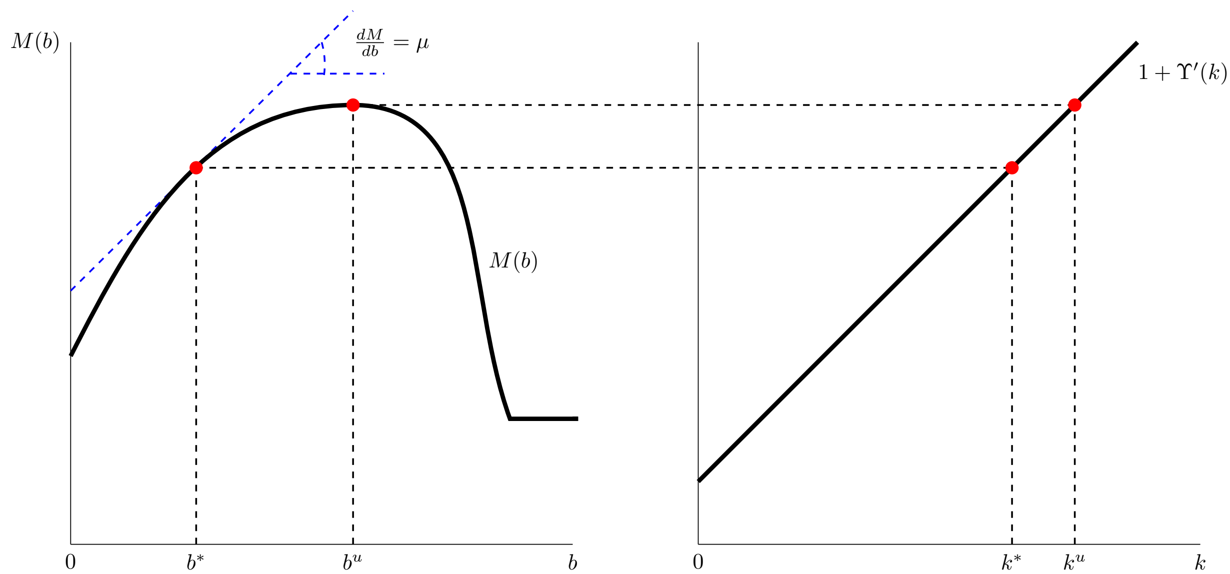

Figure 1: Sensitivity of investment to leverage constraint

Note: Figure 1 shows the optimal joint determination of leverage (left panel) and investment (right panel) in Proposition 1, illustrating also Lemma 2. This figure should be read from left to right. Changes in the leverage cap $\bar{b}$ around the laissez-faire optimum (b^u) are associated with no changes in the level of investment, since $\frac{dM}{db}(b^u) = 0$ in that case. Changes in the leverage cap away from the laissez-faire optimum (for instance around $b^\star$) induce changes in the level of investment that are increasing in the slope of $\frac{dM}{db}$ and are modulated by $\Upsilon''(k)$, which determines the slope of $1 + \Upsilon'(k)$ in the right panel.

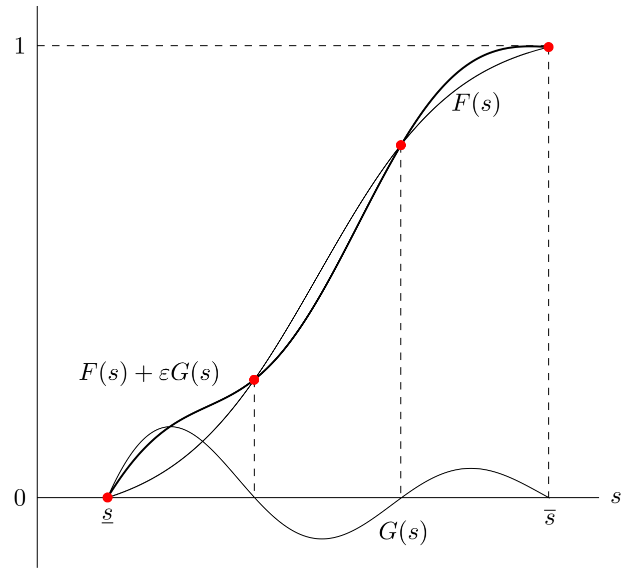

Figure 2: Perturbation of beliefs

Note: Figure 2 illustrates a perturbation/variation of beliefs, starting from the distribution of beliefs with cdf $F(s)$, in the direction of $G(s)$. Note that $G(s)$ satisfies $G(\underline{s}) = G(\overline{s}) = 0$ and is such that $F'(s) + \varepsilon G'(s) \geq 0$, $\forall s$.

increases (decreases) investment in proportion to the shadow value of the leverage constraint μ.

The Lagrange multiplier μ in the leverage constraint plays a key role in this paper. It defines the equilibrium private marginal net benefit of leverage per unit of investment in equilibrium, as shown in Equation (6). At the same time, as shown by Lemma 2, μ determines the sensitivity of investment to a change in the leverage constraint. Lemma 2 also implies that a marginal tightening of the leverage constraint around the laissez-faire outcome has no impact on investment, since $\left.\frac{dk^\star}{d\bar{b}}\right|_{\mu=0} = 0$. Figure 1 provides a graphical illustration of Lemma 1.

Next, we characterize the response of equilibrium leverage and investment to changes in beliefs. These responses are a key input into our analysis in the next section, where we study how beliefs affect the optimal regulatory policy. Because we have specified flexible, non-parametric distributions of investors' and creditors' beliefs, beliefs are infinite-dimensional objects in our analysis. We therefore characterize the responses of leverage and investments using the calculus of variations.[9]

Formally, we consider perturbations of beliefs of the form $F(s) + \varepsilon G(s)$ where $F(s)$ denotes the original cumulative distribution of s, the variation $G(s)$ represents the direction

[9]See Luenberger (1997) for a formal treatment, and Golosov et al. (2014) for a recent application of functional differentiation to optimal taxation. To our knowledge, we provide the first application of these techniques to environments with beliefs heterogeneity.

of the perturbation of beliefs, and $\varepsilon \geq 0$ is an arbitrary scalar. Figure 2 illustrates an arbitrary perturbation of $F(s)$. We use the operator δ to denote functional derivatives, as described in the following definition. For the perturbation considered here to be valid, it is necessary that $F(\cdot) + \varepsilon G(\cdot)$ remains a cdf for small enough ε.[10] Therefore, we assume throughout that the variation $G(\cdot)$ satisfies three conditions: *i)* $G(s)$ is continuous and differentiable in s, *ii)* $G(\underline{s}) = G(\overline{s}) = 0$, and *iii)* $F'(s) + \varepsilon G'(s) \geq 0$ for small enough ε, $\forall s$. The variation $G(s)$ can be interpreted as a measure of cumulative optimism relative to $F(s)$. When $G(s) < 0$, an individual thinks that the probability of returns lower than s is lower under the perturbed beliefs. For example, the perturbation we consider induces agents to become more optimistic in the sense of first-order stochastic dominance whenever $G(s) \leq 0$.

For concreteness, we write the market value $M\left(b; F^I, F^C\right)$ and the marginal value $\frac{dM}{db}\left(b; F^I, F^C\right)$ of leverage as explicit functions of creditors' and investors' beliefs. We then define variational derivatives with respect to investors' and creditors' beliefs as follows:

Definition. *[Variational derivative] The variational derivative of $M\left(b; F^I, F^C\right)$ when the perceived distribution $F^I(s)$ changes in the direction of $G^I(s)$ is denoted by $\frac{\delta M}{\delta F^I} \cdot G^I$ and is defined as*

$$\frac{\delta M}{\delta F^I} \cdot G^I = \lim_{\varepsilon \to 0} \left[\frac{M\left(b; F^I + \varepsilon G^I, F^C\right) - M\left(b; F^I, F^C\right)}{\varepsilon} \right].$$

Analogously, the variational derivative of $\frac{dM}{db}\left(b; F^I, F^C\right)$ when the perceived distribution $F^I(s)$ changes in the direction of $G^I(s)$ is denoted by $\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I} \cdot G^I$ and is defined as

$$\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I} \cdot G^I = \lim_{\varepsilon \to 0} \left[\frac{\frac{dM}{db}\left(b; F^I + \varepsilon G^I, F^C\right) - \frac{dM}{db}\left(b; F^I, F^C\right)}{\varepsilon} \right].$$

The Appendix includes the counterparts of these definitions for variations in creditors' beliefs.

These variational derivatives are key statistics for the effect of belief distortions on investor behavior. Indeed, suppose that the beliefs of agents $j \in \{I, C\}$ change in the direction $G^j(s)$. Applying the implicit function theorem to investors' first-order conditions in Proposition 1, we similarly obtain the variational derivatives of leverage and investment:

Lemma 3. *[Sensitivity of leverage and investment to beliefs] The sensitivities of equilibrium leverage and equilibrium investment to changes in investors and creditors beliefs are*

[10]Note that perturbations of the form $F(s)+\varepsilon G(s)$ are identical to perturbations of the form $(1-\varepsilon) F(s) + \varepsilon \tilde{G}(s)$, since the latter expression can be reformulated as $F(s) + \varepsilon G(s)$, where $G(s) \equiv \tilde{G}(s) - F(s)$.

respectively given by

$$\frac{\delta b^\star}{\delta F^I}\cdot G^I=\frac{\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I}\cdot G^I}{-\frac{d^2M}{db^2}},\qquad \frac{\delta b^\star}{\delta F^C}\cdot G^C=\frac{\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^C}\cdot G^C}{-\frac{d^2M}{db^2}}$$
$$\frac{\delta k^\star}{\delta F^I}\cdot G^I=\frac{\frac{\delta M}{\delta F^I}\cdot G^I}{\Upsilon''\left(k^\star\right)},\qquad \frac{\delta k^\star}{\delta F^C}\cdot G^C=\frac{\frac{\delta M}{\delta F^C}\cdot G^C}{\Upsilon''\left(k^\star\right)}.$$

These characterizations show that the same perturbation of beliefs impacts leverage and investment through different channels: Leverage changes in proportion to the variational derivative of the private marginal value $\frac{dM}{db}$, while investment changes in proportion to the variational derivative of the total valuation $M(b)$.[11] This subtle distinction will be a key driver of our normative results. We characterize the key variational derivatives next in Proposition 2.

Proposition 2. *a) [Variational derivatives: Market value] The market value $M\left(b;F^I(\cdot),F^C(\cdot)\right)$ changes in response to distortions in investors' and creditors' beliefs according to*

$$\frac{\delta M}{\delta F^I}\cdot G^I=-\beta^I\int_{s^\star(b)}^{\overline{s}}G^I(s)\,ds \tag{10}$$
$$\frac{\delta M}{\delta F^C}\cdot G^C=-\beta^C\left[(1-\phi)\,s^\star(b)\,G^C\left(s^\star(b)\right)+\phi\int_{\underline{s}}^{s^\star(b)}G^C(s)\,ds\right]. \tag{11}$$

b) [Variational derivatives: Marginal value of leverage] The marginal value of leverage $\frac{dM}{db}\left(b;F^I(\cdot),F^C(\cdot)\right)$ changes in response to distortions in investors' and creditors' beliefs according to

$$\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I}\cdot G^I=\beta^I G^I\left(s^\star(b)\right) \tag{12}$$
$$\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^C}\cdot G^C=-\beta^C G^C\left(s^\star(b)\right)\left(1+(1-\phi)\,s^\star(b)\,\frac{g^C\left(s^\star(b)\right)}{G^C\left(s^\star(b)\right)}\right), \tag{13}$$

where $g^C\left(s^\star(b)\right)=\left(G^C\right)'\left(s^\star(b)\right)$.

The upshot of studying functional derivatives is that we can consider arbitrary changes in beliefs. The general characterization in Proposition 2 shows that both the *type* and the *magnitude* of changes in beliefs are critical to understanding the behavior of leverage and investment. We obtain several useful insights from this characterization. First of all, part

[11] A first look at Equation (8) may imply that $\frac{\delta k^\star}{\delta F^I}\cdot G^I$ also depends on $\frac{\delta b^\star}{\delta F^I}\cdot G^I$, since $\frac{\delta k^\star}{\delta F^I}\cdot G^I=\frac{\frac{dM}{db}\frac{\delta b^\star}{\delta F^I}\cdot G^I+\frac{\delta M}{\delta F^I}\cdot G^I}{\Upsilon''(k^\star)}$. However, the term $\frac{dM}{db}\frac{db^\star}{dF^I}\cdot G^I$ is always 0, either because firms choose leverage optimally (and $\frac{dM}{db}=0$) or because the constraint binds (and $\frac{db^\star}{dF^I}\cdot G^I=0$).

a) of the proposition shows that only changes in some parts of the distribution of beliefs of either investors or creditors will affect the market value $M(b)$, and hence investment choices, in equilibrium. For instance, changes in beliefs that only affect investors' beliefs in solvent states $s > s^\star(b)$ will have no impact on investor's leverage choices. Creditors' cumulative belief distortions in the marginal default state $s^\star(b)$ play a key role.

Second, part b) of Proposition 2 considers the marginal value $\frac{dM(b)}{db}$ of leverage, and exposes a fundamental asymmetry between the responses of leverage to creditors' and investors' beliefs. Equations (12) and (13) suggest that leverage responds positively to cumulative distortions $G^I(s^\star(b))$ in investors' beliefs, but negatively to distortions $G^C(s^\star(b))$ in creditors' beliefs. In Section 4 below, we show that these effects become even more nuanced when there are government bailouts.

In this context, it is useful to compare distributions using a particular stochastic order. The notion that delivers unambiguous results in our model is *hazard rate dominance*.

Definition. *[Hazard rate dominance] Given two absolutely continuous distributions with cdf's $F^1(\cdot)$ and $F^2(\cdot)$, and pdf's $f^1(\cdot)$ and $f^2(\cdot)$, respectively, with support $[\underline{s}, \overline{s}]$, F^1 stochastically dominates F^2 in a hazard rate sense if*

$$\frac{f^1(s)}{1-F^1(s)} \leq \frac{f^2(s)}{1-F^2(s)}, \quad \forall s \in [\underline{s}, \overline{s}].$$

Hazard rate dominance is a natural definition of optimism. It captures the idea that optimists are increasingly optimistic over their assessments of upper-thresholds events of the form $[s, \overline{s}]$ when increasing the threshold $s \in [\underline{s}, \overline{s}]$. Formally, hazard rate dominance is equivalent to stating that $\frac{1-F^1(s)}{1-F^2(s)}$ is increasing in s. Hazard rate dominance is a stronger requirement than first-order stochastic dominance, but a weaker requirement than the monotone likelihood ratio property.[12] Our results rely on hazard-rate dominance because of the deadweight loss of default. When the deadweight of default approaches zero ($\phi \to 1$), we could derive all our results using first-order stochastic dominance.

Equation (10) shows that the market value increases when investors become more optimistic at the margin about solvent states $s \geq s^\star(b)$. Equation (11) shows that the market value also increases when creditors become more optimistic about default states $s < s^\star(b)$. They perceive that their recovery rate after default will be higher and that the marginal cost of distress will be lower, since investors will default less frequently.

Equation (12) shows that the marginal value of leverage is lower when investors become more optimistic. They perceive that they will repay their debt more often, which makes leverage costlier. Crucially, Equation (13) shows that the marginal value of leverage is

[12] F^1 stochastically dominates F^2 in a first-order sense if $F^1(s) \leq F^2(s)$, $\forall s \in [\underline{s}, \overline{s}]$.

higher when creditors become more optimistic.[13] We collect these insights in Proposition (3).

Proposition 3. *[Differential impact of optimism by investors and creditors] Optimism and pessimism in this proposition are defined in the sense of hazard rate dominance.*

a) [Equity exuberance] When investors become more optimistic, investment increases but leverage decreases. When investors become more pessimistic, investment decreases but leverage increases.

b) [Debt exuberance] When creditors become more optimistic, both investment and leverage increase. When creditors become more pessimistic, both investment and leverage decrease.

c) [Joint exuberance] When investors and creditors have common beliefs and both become more optimistic, both investment and leverage increase. When investors and creditors have common beliefs and both become more pessimistic, both investment and leverage decrease.

Proposition 3 exposes a fundamental asymmetry in the impact of optimism on equilibrium leverage and investment. Optimism on the credit supply side of the economy (creditors) is directly associated with higher leverage and investment. In this case, optimistic creditors are willing to offer credit more cheaply, which encourages investors to take on higher leverage, through a substitution effect.

By contrast, higher optimism on the credit demand side (equity investors) has more subtle implications. When investors become more optimistic about the profitability of their investment, they find it optimal to increase their equity contribution. This is a natural response: Investors find their investment very profitable, so they want to increase the contribution that they make to the investment with their own funds.

Interestingly and unexpectedly, we find that in a joint exuberance scenario in which both investors and creditors become more optimistic starting from a common belief assessment, both leverage and investment behave as in the case in which creditors become more optimistic. Equations (12) and (13) allow us to give some intuition for this result. While equity exuberance makes debt expensive at the margin, debt exuberance makes it cheap, but the debtholders value the payments in the marginal states more, because they have a higher discount factor, so with joint exuberance, debt becomes cheaper overall.

[13] In Section E.3 of the Appendix, we show that any belief perturbation $G^j(s)$ that induces more optimism in the sense of hazard-rate dominance implies that $G^j(s) \leq 0$ and $g^j(s) + G^j(s)\frac{f^j(s)}{1-F^j(s)} \leq 0$, $\forall s$. Moreover, $\frac{dM}{db} = \mu \geq 0$ implies that $1 \geq (1-\phi)\frac{s^\star(b) f^C(s^\star(b))}{1-F^C(s^\star(b))}$. It follows that the term in brackets in Equation (13) is always positive.

3 Optimal regulation

After characterizing the impact of beliefs on equilibrium leverage and investment, we are ready to study how to optimally regulate leverage, which is the ultimate goal of this paper. We adopt a flexible approach to how the government computes social welfare. That is, we assume that the government assesses the likelihood of events using a distribution $F^{C,P}(s)$ for creditors' consumption and a distribution $F^{I,P}(s)$ for investors' consumption. This approach allows us to explore many alternative combinations. For instance, a planner that respects agent's beliefs will set $F^{C,P}(s) = F^{C}(s)$ and $F^{I,P}(s) = F^{I}(s)$. Alternatively, a planner who computes social welfare using the true distribution of investment return will set $F^{C,P}(s) = F^{I,P}(s) = F(s)$.

Given this approach, we can now characterize an optimal paternalistic policy, which takes into account the fact that agents makes decisions under their own potentially distorted beliefs, but evaluates the consequences of these decisions using the planner's beliefs, which could be perceived as correct. Assuming that the government has correct beliefs places a high burden of foresight on the government, but has the advantage of bringing out the underlying economic effects most cleanly.

3.1 Planner's problem

The first step is to formulate the planner's objective. Lemma 4 formally characterizes social welfare from the perspective of a utilitarian planner as a function of the leverage cap $\bar{b}$.

Lemma 4. *[Planner's problem] The planner's problem can be expressed as*

$$\max_{\bar{b}} W\left(b^{\star}\left(\bar{b}\right), k^{\star}\left(\bar{b}\right)\right),$$

where social welfare $W(b,k)$ is given by

$$W(b,k) = \left[M^{P}(b) - 1\right]k - \Upsilon(k),$$

and where $M^{P}(b)$ denotes the present-value of payoffs under the planner's beliefs

$$M^{P}(b) = \beta^{C}\int_{\underline{s}}^{\bar{s}} s dF^{C,P}(s) - \int_{s^{\star}(b)}^{\bar{s}} (s-b)\left(\beta^{C} dF^{C,P}(s) - \beta^{I} dF^{I,P}(s)\right) - \beta^{C}\int_{\underline{s}}^{s^{\star}(b)} (1-\phi)\, s dF^{C,P}(s).$$

The planner's objective mimics the objective faced by investors when deciding how much to borrow and invest — see Equation (2) — after incorporating the planner's beliefs. This result is intuitive but not obvious, since the welfare of investors as perceived by the planner depends on the actual beliefs of creditors. Note that whenever leverage regulation is binding, social welfare depends on investors' and creditors' beliefs only through the investment choice $k^{\star}$, since in that case $b^{\star}\left(\bar{b}\right)=\bar{b}$ is directly controlled by the planner.

3.2 Local welfare effects

We focus our attention on the *local welfare effect* $\frac{dW}{d\bar{b}}$ of varying the leverage cap. One can take this analysis further by studying under what conditions the welfare function is quasi-concave in $\bar{b}$. Under such conditions, one would be able to translate our results into explicit comparative statics on the optimal policy $b^{\star}$, which would be the solution to $\frac{dW}{d\bar{b}}=0$. We choose to focus on local effects since they allow us show the relevant economic effects more clearly.

Proposition 4. *[Local Welfare Effect] The marginal welfare impact of increasing the leverage cap, whenever the leverage cap is binding, is*

$$\frac{dW}{d\bar{b}}=\underbrace{\frac{dM^{P}\left(\bar{b}\right)}{d\bar{b}}k^{\star}\left(\bar{b}\right)}_{\text{Inframarginal Effect}}+\underbrace{\left[M^{P}\left(\bar{b}\right)-M\left(\bar{b}\right)\right]\frac{dk^{\star}\left(\bar{b}\right)}{d\bar{b}}}_{\text{Incentive Effect}}, \tag{14}$$

where we provide an explicit characterization of each of its elements in the Appendix.

Proposition 4 shows that knowledge of $M^{P}\left(b\right)$, $M\left(b\right)$, and $k^{\star}\left(\bar{b}\right)$ and their derivatives is sufficient to determine whether it is desirable to increase or decrease the leverage cap. We refer to the first term in Equation (14) as the inframarginal effect. This term captures how varying the leverage cap modifies the planner's valuation of the pre-existing investment at the margin. We refer to the second term in Equation (14) as the incentive effect. This term captures the investment response associated with a change in $\bar{b}$. As shown in Lemma 2, this term is weakly positive.

Furthermore, Proposition 4 implies a simple test for whether the local welfare effect of raising the leverage cap is positive:

$$\frac{dW}{d\bar{b}}>0\Leftrightarrow\frac{d\ln M^{P}\left(\bar{b}\right)}{db}+\left[\frac{M^{P}\left(b\right)-M\left(b\right)}{M^{P}\left(b\right)}\right]\frac{d\ln k\left(\bar{b}\right)}{d\bar{b}}>0 \tag{15}$$

This characterization reveals three sufficient statistics that need to be measured empirically to determine whether leverage regulation should become stricter or more relaxed. The first corresponds to the marginal social benefit $\frac{d\ln M^{P}\left(\bar{b}\right)}{db}$ of leverage. The second is the wedge $\frac{M^{P}(b)-M(b)}{M^{P}(b)}$, which measures the proportional difference between the planner's and investors' perception of the present-value of investment. This wedge is perhaps the most challenging statistic to measure because it requires a direct assessment of belief differences. The third statistic is the semi-elasticity $\frac{d \ln k(\bar{b})}{d\bar{b}}$ of investment to leverage requirements, which equals the coefficient in a standard regression of log investment on exogenous variation in leverage limits.

If one starts from the laissez-faire allocation, the incentive effect vanishes and $\frac{d \ln k(\bar{b})}{d\bar{b}} = 0$ (see Figure 1 for an illustration). In that case, the only sufficient statistic that characterizes the local effect is $\frac{d \ln M^P(\bar{b})}{db}$. This is an interesting observation, since it does not require the planner to know the beliefs of investors or creditors. It is sufficient for the planner to form an assessment over what the value of $\frac{dM^P(\bar{b})}{db}$ is. Note also that the rationale for regulation in this model is precisely the presence of distorted beliefs: When the planner and the agents have the same set of beliefs, the incentive effect vanishes — since $M^P\left(\bar{b}\right) = M\left(\bar{b}\right)$ — and the inframarginal effect is always positive whenever the constraint binds, so it is optimal to never set a leverage cap.

These observations highlight an independent contribution of this paper. We characterize the sufficient statistics that are needed to capital optimal leverage caps. As we will show in Section 4, similar statistics characterize optimal regulation in a model where there are additional differences between private and social incentives due to government bailouts. In that section, there is a case for regulation even if beliefs are not distorted. Hence, the statistics we present also inform on optimal regulation in rational environments.

3.3 The impact of optimism/pessimism on leverage regulation

This subsection introduces the main results of the paper. It characterizes how a change in beliefs by investors or creditors modifies the form of the optimal financial regulatory policy. As is familiar from the literature on monotone comparative statics (Milgrom and Shannon, 1994), the effects of beliefs on optimal policy is characterized by the supermodularity of welfare. Optimal leverage caps increase when the local welfare effect $\frac{dW}{d\bar{b}}$ has a positive derivative with respect to beliefs. In the context of our flexible specification, beliefs are infinite-dimensional. Therefore, we consider the *variational* derivative of $\frac{dW}{d\bar{b}}$ with respect to beliefs in Proposition 5.

Proposition 5. *[Impact of beliefs on optimal regulation: general characterization] The change in the marginal welfare impact of increasing the leverage cap, whenever the leverage cap is binding, in response to a change in beliefs by either investors or creditors, $j = \{I, C\}$, is given by*

$$\frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j = \left[\frac{dM^P\left(\bar{b}\right)}{db} - \frac{dM\left(\bar{b}\right)}{db}\right] \left[\frac{\delta k^\star\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right] + \left[M^P\left(\bar{b}\right) - M\left(\bar{b}\right)\right] \left[\frac{\delta \frac{dk^\star\left(\bar{b}\right)}{db}}{\delta F^j} \cdot G^j\right] \quad (16)$$

Proposition 5 permits a clearer assessment of how, and why, optimal capital regulation changes when investors' and/or borrowers' beliefs change. Belief distortions affect optimal policy through two channels, namely, through the change in optimal investment $k^{\star}\left(\bar{b}\right)$ and the change in the sensitivity $\frac{dk^{\star}\left(\bar{b}\right)}{db}$ of investment to policy. These effects correspond to the inframarginal and incentive effects in our analysis above. Intuitively, when deciding how to respond to a new belief distortion with leverage policy, the planner must assess *i)* the extent to which the distortion affects investment behavior and the desirability of regulation, and *ii)* the extent to which it affects the sensitivity or effectiveness of regulation.

The comparative statics in Lemma 3 imply that the variational derivatives of $k^{\star}(b)$ and $\frac{dk^{\star}\left(\bar{b}\right)}{db}$ are directly related to the variational derivatives of the market value $M(b)$ and marginal market value $\frac{dM}{db}$. Therefore, the effects of investor beliefs on optimal regulation inherit the nuanced patterns associated with $\frac{\delta M\left(\bar{b}\right)}{\delta F^{j}} \cdot G^{j}$ and $\frac{\delta \frac{dM}{db}}{\delta F^{j}} \cdot G^{j}$, which we characterized in Section 2. As implied by our detailed discussion of Propositions 2 and 3, the type of belief distortion is important, and in particular cumulative distortions at the default boundary, $G^{j}\left(s^{\star}(b)\right)$, play a special role. In particular, the type of distortion becomes crucial, and there is a fundamental asymmetry between investors and creditors. It is particularly instructive in this context to consider the case of quadratic adjustment costs, $\Upsilon'''(\cdot) = 0$, in which Equation (16) simplifies to

$$\frac{\delta \frac{dW}{db}}{\delta F^{j}} \cdot G^{j} = \varphi \cdot \left(\left[\frac{dM^{P}\left(\bar{b}\right)}{db} - \frac{dM\left(\bar{b}\right)}{db}\right]\left[\frac{\delta M\left(\bar{b}\right)}{\delta F^{j}} \cdot G^{j}\right] + \left[M^{P}\left(\bar{b}\right) - M\left(\bar{b}\right)\right]\left[\frac{\delta \frac{dM}{db}}{\delta F^{j}} \cdot G^{j}\right]\right),$$

We can now directly leverage the results in Proposition 3 to characterize the effect of belief distortions on regulation. We consider three scenarios:

Proposition 6. *[Impact of beliefs on optimal regulation: Specific scenarios] In this proposition, we assume that adjustment costs are quadratic.*

a) <u>Equity exuberance:</u> Assume that creditors and the planner share a common belief $F^{C}(s) = F^{C,P}(s) = F^{I,P}(s)$, *and investors' beliefs are more optimistic than the planner's beliefs in a hazard rate sense. Then, increased optimism by investors implies* $\frac{\delta \frac{dW}{db}}{\delta F^{I}} \cdot G^{I} > 0$. *Hence, it is never optimal to impose a binding leverage cap.*

b) <u>Debt exuberance:</u> Assume that investors and the planner share a common belief $F^{I}(s) = F^{I,P}(s) = F^{C,P}(s)$, *and creditors' beliefs are more optimistic than the planner's beliefs in a hazard rate sense. Then, increased optimism by investors implies* $\frac{\delta \frac{dW}{db}}{\delta F^{I}} \cdot G^{I} < 0$. *Hence, the optimal leverage cap is binding and decreasing in optimism.*

c) <u>Joint exuberance:</u> Assume that creditors and investors share a common belief $F^{C}(s) = F^{I}(s)$ *that is more optimistic than the planner's belief in a hazard rate sense. Then, as in the debt exuberance scenario, increased optimism by investors and creditors implies* $\frac{\delta \frac{dW}{db}}{\delta F^{I}} \cdot G^{I} < 0$.

Proposition 6 shows a clear distinction between the effects of equity and debt exuberance. In the case of equity exuberance, there are two effects. First, investors consider creditors' beliefs to be excessively pessimistic, which leads them to take too little leverage (i.e., issue too much equity) from a social perspective. Thus, the inframarginal effect increases the social benefit of encouraging leverage. Second, the incentive effect becomes weaker with exuberance due to the reduced sensitivity of investment to leverage. Both effects imply that exuberance increases the marginal benefit of permitting leverage. Starting from a case without belief distortions, where there is no rationale for a binding leverage cap, we therefore find that equity exuberance only serves to make a binding cap *less* desirable. We conclude that it is never optimal to impose a binding cap. Instead, it would be optimal to impose a binding leverage floor, although this is not a policy we have considered in our analysis.

The recent experience of Hertz is a clear example of this scenario.[14] In April of 2020, there seemed to be investors willing to purchase Hertz stock even though the company has declared bankruptcy, and it is unlikely that equityholders would receive any funds at all. Hertz management, seeking to maximize the firm value, decided to sell shares in the open market. The regulator (in this case, the SEC) vetoed the issuance, which can be interpreted as a cap on equity (a debt floor).

In the case of debt exuberance, both the inframarginal and incentive effects are reversed. First of all, investors consider debt to be overvalued, which leads them to take too much leverage. Moreover, the incentive effect becomes stronger due to the increased sensitivity of investment to leverage. Both effects work in the same direction, and imply that a binding leverage cap is optimal. Finally, we show that the case of joint exuberance leads to the same qualitative result as the case of debt exuberance. As discussed in the context of Proposition 3, this is because the valuation of marginal default states by creditors is more important than investors' valuation, so that debt exuberance determines the sign of the key sufficient statistics. As a result, joint exuberance also implies a case for a binding leverage cap, albeit a lower one than in the case of pure debt exuberance.

4 Government bailouts

Proposition 6 implies that equity exuberance motivates higher leverage caps. In this section, we show that this result can be overturned in the presence of an additional wedge between private and social incentives. More broadly, we show that bailouts are important because their interaction with beliefs is highly nonlinear.

[14] See https://www.wsj.com/articles/hertz-sold-29-million-in-stock-before-sec-stepped-in-11597100128.

4.1 Environment

In particular, we consider an extension of our model with government bailouts. We assume that at date 1, after the state s is realized, the government makes a transfer $t(b,s)$ to investors. The funds for this transfer are raised using a tax $(1+\kappa)t(b,s)$ on creditors, where $\kappa > 0$ measures the deadweight loss associated with taxation. Investors decide whether to default. If investors default, creditors seize all of the investors' resources — including government transfers — and receive $\phi s + t(b,s)$ per unit of investment. We assume throughout that the value of banks' assets including the bailout $s + t(b,s)$ is increasing in s. This implies the existence of a unique threshold $s^{\star}(b)$ such that investors default if $s < s^{\star}(b)$ and repay otherwise.

We define the expected fiscal burden perceived by the planner, per unit of investment, as

$$\gamma(b) = (1+\kappa)\,\beta^{C} \int_{\underline{s}}^{\bar{s}} t(b,s)\,dF^{C,P}(s).$$

This burden assumes the role of an additional wedge between private and public incentives.

4.2 Local welfare effects and bailouts

We can characterize the effect of belief distortions on leverage regulation with bailouts by extending Proposition 5:

Proposition 7. *[Impact of beliefs on optimal regulation with bailouts] The change in the marginal welfare impact of increasing the leverage cap, whenever the leverage cap is binding, in response to a change in beliefs by either investors or creditors, $j = \{I, C\}$, is given by*

$$\frac{\delta \frac{dW}{db}}{\delta F^{j}} \cdot G^{j} = \left[\frac{dM^{P}\left(\bar{b}\right)}{db} - \frac{d\gamma(b)}{db} - \frac{dM\left(\bar{b}\right)}{db}\right]\left[\frac{\delta k^{\star}\left(\bar{b}\right)}{\delta F^{j}} \cdot G^{j}\right] + \left[M^{P}\left(\bar{b}\right) - \gamma(b) - M\left(\bar{b}\right)\right]\left[\frac{\delta \frac{dk^{\star}\left(\bar{b}\right)}{db}}{\delta F^{j}} \cdot G^{j}\right] \tag{17}$$

where the adjusted valuation functions $M^{P}(b)$ and $M(b)$ are characterized in the Appendix.

The first term, reflecting the inframarginal effect of raising the leverage cap, now includes the increase in the fiscal burden $\frac{d\gamma(b)}{db}$. The second term, reflecting the incentive effect, scales with the total wedge between social and private incentives, which now includes the level of the fiscal burden $\gamma(b)$ per unit of investment.

This result implies that the consequences of equity exuberance are different when there are bailouts. Proposition 6 in the baseline model shows that equity exuberance always decreases the incentive to regulate, in part because the inframarginal effect of raising leverage caps is always positive. In the model with bailouts, by contrast, the inframarginal effect can change sign due to the presence of bailouts, because $\frac{d\gamma(b)}{db} > 0$. Therefore, the type as well as the extent of equity exuberance becomes important when there are bailouts. We return to this point below, where we consider a concrete example of the bailout policy $t(b, s)$.

In Proposition 8, which is the counterpart of Proposition 2, we further analyze the general interaction of belief distortions and bailouts by characterizing the key variational derivatives:

Proposition 8. *a) [Variational derivatives: Market value with bailouts] The market value $M\left(b; F^I(\cdot), F^C(\cdot)\right)$ changes in response to distortions in investors' and creditors' beliefs according to*

$$\frac{\delta M}{\delta F^I} \cdot G^I = -\beta^I \int_{s^\star(b)}^{\bar{s}} \left(1 + \frac{\partial t(b,s)}{\partial s}\right) G^I(s)\, ds \tag{18}$$

$$\frac{\delta M}{\delta F^C} \cdot G^C = -\beta^C \left[(1-\phi)\, s^\star(b)\, G^C(s^\star(b)) + \int_{\underline{s}}^{s^\star(b)} \left(\phi + \frac{\partial t(b,s)}{\partial s}\right) G^C(s)\, ds\right]. \tag{19}$$

b) [Variational derivatives: Marginal value of leverage with bailouts] The marginal value of leverage $\frac{dM}{db}\left(b; F^I(\cdot), F^C(\cdot)\right)$ changes in response to distortions in investors' and creditors' beliefs according to

$$\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I} \cdot G^I = -\beta^I \left[\int_{s^\star}^{\bar{s}} dG^I(s) + \frac{\partial t(b,s^\star)}{\partial b} G^I(s^\star) + \int_{s^\star}^{\bar{s}} \frac{\partial^2 t(b,s)}{\partial b \partial s} G^I(s)\, ds\right] \tag{20}$$

$$\begin{aligned}\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^C} \cdot G^C = -\beta^C \Bigg[& G^C(s^\star(b)) \frac{\partial s^\star(b)}{\partial b} \left(1 + \frac{\partial t(b,s^\star(b))}{\partial s} + (1-\phi)\, s^\star(b) \frac{g^C(s^\star(b))}{G^C(s^\star(b))}\right) \\ & + \int_0^{s^\star(b)} \frac{\partial^2 t(b,s)}{\partial b \partial s} G^C(s)\, ds\Bigg]\end{aligned} \tag{21}$$

Proposition 8 conveys several insights. Part a) shows that, if bailouts satisfy $\frac{\partial t(b,s)}{\partial s} \leq 0$, then the presence of bailouts attenuates the sensitivity of the market valuation $M(b)$ to belief distortions $G^I(s)$ and $G^C(s)$. Moreover, if bailouts are convex in s, so that $\frac{\partial t(b,s)}{\partial s}$ is larger in absolute value for low s, then the attenuation effect is skewed towards belief distortions in bad states. Intuitively, bailouts imply that creditors' beliefs become less important for market valuation.

Part b) shows that, if bailouts also satisfy $\frac{\partial t(b,s)}{\partial b} \geq 0$, then the effect of belief distortions in the marginal default state $s^\star(b)$ on the marginal valuation $\frac{dM}{db}$ is attenuated towards zero by the presence of bailouts. In addition, both variational derivatives of $\frac{dM}{db}$ contain a term with the sign of $-\frac{\partial^2 t(b,s)}{\partial b \partial s} G^j(s)$ for $j \in \{I, C\}$. These terms arise because belief distortions affect investors' strategic incentive to take leverage in order to increase bailouts. If the strategic incentive $\frac{\partial t(b,s)}{\partial b}$ is decreasing in s,[15] then optimism increases $\frac{dM}{db}$. This effect of bailouts strengthens the negative effect of equity optimism on incentives to take leverage, but weakens the positive effect of debt optimism.

4.3 Too-Big-To-Fail scenario

It is instructive to consider the common special case in which bailouts are perfectly targeted towards avoiding bankruptcy, so that $t(b,s) = \max\{b-s,0\}$. This scenario corresponds to interpreting the investor as a bank that is "too big to fail" (TBTF). It illustrates clearly that bailouts attenuate the role of creditors' beliefs, and that they can overturn our normative conclusions in an equity exuberance scenario.

Example. *[Too-Big-To-Fail scenario.] Assume that bailouts satisfy* $t(b,s) = \max\{b-s,0\}$. *In this case, the bank's market valuation* $M(b)$ *reduces to*

$$M(b) = \beta^I \int_{s^\star(b)}^{\overline{s}} (s-b)\, dF^I(s) + \beta^C b,$$

where $s^\star(b) = b$.

In this limiting case, there is no risk of default, so that the valuation of debt is independent of creditors' beliefs F^C. We can therefore focus on belief distortions among investors. For an equity exuberance scenario, we can simplify (17) to obtain

$$\frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j \leq 0 \Leftrightarrow \frac{G^I(b)}{\int_{s^\star(b)}^{\overline{s}} G^I(s)\, ds} \leq \frac{\beta^C(1+\kappa) - \beta^I F^{I,P}(b) + \beta^I F^I(b)}{\gamma(b) + \beta^I \int_{s^\star(b)}^{\overline{s}} \left(F^{I,P}(s) - F^I(s)\right) ds}. \tag{22}$$

In the case where the variational derivative is positive, our result in the baseline model is overturned, and equity exuberance increases incentives to constrain leverage. Indeed, perfect bailouts are guaranteed to change the sign of the inframarginal term in Equation (17), so that the planner always has an additional inherent incentive to discourage leverage.[16]

Equation (22) shows that the type of belief distortion is key when determining the strength of this effect. Indeed, the incentive to cap leverage dominates when the "downside" belief distortion in marginal bailout states $G^I(b)$ is small relative to the overall "upside" distortion in solvent states $\int_b^{\overline{s}} G^I(s)\, ds$. Intuitively, the inframarginal term scales with the level of investment $k^\star(b)$, which in this scenario is determined purely by investors' beliefs

[15] Bailouts are often modeled as a convex function of the shortfall $b-s$ of asset values from debt obligations. This directly implies $\frac{\partial^2 t(b,s)}{\partial b \partial s} \leq 0$.

[16] In the TBTF scenario, we have $\frac{dM^P}{db} - \frac{d\gamma}{db} - \frac{dM}{db} = -\left(\left(\beta^C(1+\kappa) - \beta^I\right) F^{I,P}(b) + \beta^I F^I(b)\right) < 0$.

about solvent states. Therefore, large upside distortions generate a strong incentive to constrain leverage. By contrast, belief distortions in marginal states mostly result in a decreased sensitivity $\frac{dk^\star(b)}{db}$ of investment to leverage regulation. Thus, large downside distortions make regulation less attractive at the margin.

5 Monetary policy

In this section, we return to our baseline model without bailouts and introduce a reduced-form treatment of monetary policy. The natural interest rate in debt markets in our model is $r^\star = \frac{1-\beta^C}{\beta^C}$. The government can set the interest rate to $r \neq r^\star$ at a cost.[17] Deviations from the natural rate incur a deadweight loss $\mathcal{L}(r) \geq 0$, which is a convex function of r.

We focus on the welfare effect of interest rate policy when beliefs are distorted. We first characterize the optimization problem faced by investors.

Lemma 5. *[Investors' problem with active monetary policy] In the model with monetary policy, investors solve*

$$\max V(\bar{b}, r) = \max_{b,k} \left[M(b,r) - 1\right] k - c(k)$$
$$s.t.\ \ b \leq \bar{b} \qquad\qquad (\mu),$$

where $M(b,r)$ denotes the market value of equity and debt per unit of investment

$$M(b,r) = \underbrace{\beta^I \int_{s^\star(b)}^{\bar{s}} (s-b)\, dF^I(s)}_{equity} + \underbrace{\beta(r) \left(\int_{s^\star(b)}^{\bar{s}} b dF^C(s) + \phi \int_{\underline{s}}^{s^\star(b)} s dF^C(s) \right)}_{debt},$$

and where $\beta(r) \equiv \frac{1}{1+r}$ is the discount factor used to price debt when the interest rate is r.

The problem in Lemma 5 is the same as their problem in the baseline model except for the valuation function $M(b,r)$. The value of equity and debt depends on monetary policy, because creditors use the discount factor $\beta(r) = \frac{1}{1+r}$ to value debt. Throughout this section, we focus on the case where $\beta(r) < \beta^I$, so that investors remain natural borrowers and the equity is priced using β^I.

We write $b^\star\left(\bar{b}, r\right)$ and $k^\star\left(\bar{b}, r\right)$ for investors' optimal choices of leverage and investment as a function of the leverage cap $\bar{b}$ and the interest rate r. Next, we derive the planner's problem.

[17] A simple micro-foundation for this distortion is that the government can tax or subsidize a risk-free storage technology that returns $r^\star$ in the absence of taxation (e.g., Farhi and Tirole, 2012).

Lemma 6. *[Planner's problem with active monetary policy] The planner's problem can be expressed as*

$$\max_{\bar{b},r} W\left(b^{\star}\left(\bar{b},r\right),k^{\star}\left(\bar{b},r\right),r\right),$$

where social welfare is given by

$$W(b,k,r)=\left[M^{P}\left(b\right)-1\right]k-\Upsilon\left(k\right)-\mathcal{L}\left(r\right),$$

and where $M^{P}\left(b\right)$, as defined in Lemma 4, denotes the present-value of payoffs under the planner's beliefs, which is independent of r.

There are two differences between this problem and the planner's problem in Lemma 4. First, the planner realizes that investors' optimal choices of leverage b and investment k are driven by both leverage caps and the interest rate. Second, the welfare function $W\left(b,k,r\right)$ is adjusted for the deadweight cost of monetary distortions. However, the function $M^{P}\left(b\right)$ used by the planner to value the payoff from investments is identical to the baseline model, and does not depend on monetary policy. This arises because monetary policy operates by changing the price of debt at date 0, which is a welfare-neutral transfer in our model.

Relying on the characterizations in Lemmas 5 and 6, we analyze the welfare effect $\frac{dW}{dr}$ of raising interest rates and its dependence of beliefs. We focus on the effects of beliefs in the equity exuberance and debt exuberance scenarios defined in Proposition 6. A full variational characterization of these effects in the Online Appendix.

Proposition 9. *[Local welfare effect of monetary policy and beliefs] The marginal welfare impact of increasing the interest rate is*

$$\frac{dW}{dr}=\left[M^{P}\left(b\right)-M\left(b,r\right)\right]\frac{dk^{\star}\left(\bar{b},r\right)}{dr}-\mathcal{L}'\left(r\right)\tag{23}$$

where $\frac{dk^{\star}(\bar{b},r)}{dr}=\frac{1}{c''\left(k^{\star}\left(\bar{b},r\right)\right)}\frac{dM}{dr}\left(b,r\right)<0$. In both an equity exuberance scenario and a debt exuberance scenario, increased optimism by investors and borrowers is associated with higher incentives to raise interest rates.

Proposition 9 shows that the welfare effect of monetary policy depends on the difference $M^{P}\left(b\right)-M\left(b,r\right)$ between the planner's and investors' valuations of investment, as well as the derivative of investment to interest rates. This derivative is always negative because an increase in r lowers bond prices. The proposition further describes the role of beliefs in the welfare effect of monetary policy. Unlike in Proposition 5, there is no inframarginal effect associated with raising the interest rate in our model because debt prices are welfare-neutral transfers. Accordingly, Equation (23) contains only an incentive effect.

Increased optimism in either a debt or equity exuberance scenario implies that the incentive effect becomes stronger. This is the result of two forces. First, increased optimism implies that the valuation wedge $M^P(b) - M(b, r)$ becomes more negative. Second, increased optimism implies that the effect of r on bond prices is stronger when bond prices are elevated, so that investment becomes more sensitive to monetary policy. Both forces increase the planner's incentive to raise interest rates.

This result stands in contrast to the more nuanced effect of belief distortions on leverage regulation. A central reason for this difference is that optimistic investors become *more* sensitive to contractionary monetary policy, but tend to become *less* sensitive to leverage caps. Therefore, monetary policy is a natural substitute when optimism blunts the effectiveness of leverage regulation.

6 Imperfectly Targeted Paternalism

Our analysis of paternalist policy makes a strong informational assumption, namely, that the government perfectly observes the beliefs $F^I(.)$ and $F^C(.)$ that investors and creditors use to evaluate the returns s to investment. In this section, we study an extension of the model where paternalism is imperfectly targeted, in the sense that the planner faces uncertainty about random sentiments, which in turn drive the beliefs of investors and creditors. For simplicity, we concentrate on the baseline model without bailouts and monetary policy. Formally, we introduce a random "sentiment" variable $\xi \in [\underline{\xi}, \bar{\xi}]$ with distribution $H(\xi)$, which indexes market sentiment about investment returns and induces beliefs $F^I(s; \xi)$ and $F^C(s; \xi)$ among investors and creditors. By contrast, the planner's beliefs $F^{I,P}(s)$ and $F^{C,P}(s)$ are independent of sentiment.[18]

The timing of events at date 0 is as follows: First, the planner sets the leverage cap $\bar{b}$ before observing the realization of sentiment ξ. Second, sentiment ξ is realized and observed by investors and creditors. Third, investors choose leverage b and investment k.

In this environment, investors' problem is exactly as in Lemma 1, except for the fact that the market valuation $M(b; \xi)$ becomes a function of sentiments:

Lemma 7. *[Investors' problem with sentiments] Investors solve the following problem to decide their optimal investment and leverage choices at date* 0 *when sentiment is* ξ*:*

$$V\left(\bar{b}; \xi\right) = \max_{b,k} \left[M(b; \xi) - 1\right] k - \Upsilon(k) \tag{24}$$

$$s.t. \quad b \leq \bar{b} \qquad\qquad (\mu(\xi)), \tag{25}$$

[18] Alternatively, one can consider an environment where the planner's beliefs also respond to sentiment. By continuity, our qualitative results below all go through in this context as long as the relative responsiveness of investors' and creditors' beliefs to sentiment is sufficiently large relative to the planners'.

where $\mu(\xi)$ denotes the Lagrange multiplier on the leverage constraint imposed by the government when sentiment is ξ, and $M(b;\xi)$ is given by

$$M(b;\xi) = \underbrace{\beta^I \int_{s^\star(b)}^{\bar{s}} (s-b)\, dF^I(s;\xi)}_{equity} + \underbrace{\beta^C \left(\int_{s^\star(b)}^{\bar{s}} b dF^C + \phi \int_{\underline{s}}^{s^\star(b)} s dF^C(s;\xi) \right)}_{debt} \tag{26}$$

We let $k^\star\left(\bar{b};\xi\right)$ and $b^\star\left(\bar{b};\xi\right)$ denote investors' optimal choice of investment k in this problem. Repeating the steps leading to Lemma 4 and Proposition 4 in the baseline model, we find that the planner's problem with sentiments is

$$\max_{\bar{b},r} \int_{\underline{\xi}}^{\bar{\xi}} W\left(b^\star\left(\bar{b};\xi\right), k^\star\left(\bar{b};\xi\right)\right) dH(\xi) \equiv E\left[W\left(b^\star\left(\bar{b};\xi\right), k^\star\left(\bar{b};\xi\right)\right)\right], \tag{27}$$

where the social welfare function is again given by

$$W(b,k) = \left[M^P(b) - 1\right] k - \Upsilon(k),$$

with the planner's valuation $M^p(b)$ of investment defined as in Lemma 4. Notice that $W(b,k)$ is independent of sentiments ξ since they do not affect the planner's beliefs. Hence, the only source of randomness in welfare stems from the fact that firms' choices of leverage and investment depend on ξ.

We now characterize the *ex ante* local welfare effect $E\left[\frac{dW}{d\bar{b}}\right]$ of leverage regulation. In order to isolate the impact of imperfect targeting, we compare the ex ante effect to a benchmark with perfect targeting. The benchmark is defined as the local welfare effect $\frac{dW^0}{d\bar{b}}$ that would arise if investors and creditors held their average beliefs with probability one.[19]

Proposition 10. *[Local Welfare Effect with Imperfect Targeting] In this proposition, we assume that investment costs are quadratic with $\Upsilon(k) = \frac{k^2}{2\varphi}$. The expected marginal welfare impact of increasing the leverage cap is*

$$E\left[\frac{dW}{d\bar{b}}\right] = \frac{dW^0}{d\bar{b}} - \varphi \cdot Cov\left[M\left(\bar{b};\xi\right), \frac{dM\left(\bar{b};\xi\right)}{d\bar{b}}\right] - \int_{\{\xi:\mu(\xi)=0\}} \frac{d\hat{W}\left(\bar{b};\xi\right)}{d\bar{b}} dH(\xi), \tag{28}$$

where $\hat{W}\left(\bar{b};\xi\right)$, defined in the appendix, is the hypothetical level of welfare obtained by en-

[19]Formally, the average beliefs are defined as $F^j(s) = \int_{\underline{\xi}}^{\bar{\xi}} F^j(s;\xi)\, dH(\xi)$ for $j \in \{I,H\}$. The benchmark value of the local welfare effect $\frac{dW^0}{d\bar{b}}$ is obtained by evaluating the expression in Proposition 4 at the average belief.

forcing leverage $b = \bar{b}$ in states where the leverage cap is not binding.

Equation (28) shows two differences between leverage regulation with perfect and imperfect targeting. First, the expected welfare effect of raising the leverage cap is reduced by the covariance between market valuations and investment sensitivity. Intuitively, when this covariance is positive, states of the world with overinvestment (high $M\left(\bar{b};\xi\right)$) coincide with states where investment is sensitive to leverage regulation (high $\frac{dM\left(\bar{b};\xi\right)}{d\bar{b}}$).[20] In that case, it is particularly valuable *ex ante* to push for lower investment by lowering the leverage cap. Second, the leverage constraint may not always be binding if there are realizations of ξ for which the associated Lagrange multiplier is $\mu(\xi) = 0$. If the (hypothetical) benefit of enforcing more leverage in these states would be positive, then the *ex ante* benefit of raising $\bar{b}$ is less than $\frac{dW^0}{d\bar{b}}$, which is reflected in the third term in (28).

These effects have ambiguous implications for optimal policy. For example, consider the case where ξ indexes the optimism of equity investors in a hazard rate sense, while creditors' beliefs agree with the planner. Equation (28) in this case implies two counteracting effets. On one hand, our analysis above implies that the covariance between valuations and sensitivity in (28) is negative, which increases $E\left[\frac{dW}{d\bar{b}}\right]$ and decreases the planner's incentive to constrain leverage. On the other hand, Proposition 3 implies that incentives to take leverage are decreasing in ξ, so that the constraint is slack in optimistic states. This effect decreases $E\left[\frac{dW}{d\bar{b}}\right]$ because (by Proposition 6) the welfare benefit of more leverage is greatest when investors are optimstic.

However, we note that these effects become unambiguous when the leverage constraint always binds.[21]

Proposition 11. *[Welfare effects with imperfect targeting: Binding constraints] In this proposition, we assume that adjustment costs are quadratic. We fix a value of $\bar{b}$ for which the leverage constraint $b \leq \bar{b}$ binds for all realizations ξ of sentiment.*

a) Equity sentiments: Assume that creditors and the planner share a common belief regardless of sentiments $F^C(s;\xi) = F^{C,P}(s) = F^{I,P}(s)$, $\forall\xi$, and investors' beliefs $F^I(s;\xi)$ exhibit increasing optimism in ξ in a hazard rate sense. Then, $E\left[\frac{dW}{d\bar{b}}\right]$ is higher than in the benchmark with perfect targeting.

b) Debt sentiments: Assume that investors and the planner share a common belief regardless of sentiments $F^I(s;\xi) = F^{C,P}(s) = F^{I,P}(s)$, $\forall\xi$, and creditors' beliefs $F^C(s;\xi)$ exhibit increasing optimism in ξ in a hazard rate sense. Then, $E\left[\frac{dW}{d\bar{b}}\right]$ is lower than in the benchmark with perfect targeting.

[20] Recall that, with quadratic costs, Equation (8) implies that the sensitivity of investment to leverage regulation is $\frac{dk^\star\left(\bar{b};\xi\right)}{d\bar{b}} = \varphi\frac{dM\left(\bar{b};\xi\right)}{d\bar{b}}$.

[21] Equivalently, the characterization below applies to the case where the planner imposes an equality constraint $b = \bar{b}$ on investors instead of the cap $b \leq \bar{b}$ that we have considered.

With binding constraints, the covariance in (28) determines welfare effects with imperfect targeting. In the case of equity sentiments, the covariance is negative as argued above. In the case of debt sentiments, the covariance is positive because market valuations and incentives to take leverage are both increasing in creditors' optimism. Thus, relaxing paternalism has meaningful effects on optimal policy, which depend on the type of belief distortion.

7 Conclusion

This paper provides a systematic analysis of financial and monetary policy in environments in which equity investors and creditors may have distorted beliefs. We show that the optimal policy response to belief distortions depends on the type as well as the extent of exuberance, and it is not generally true that regulators should lean against the wind by tightening leverage caps in response to optimism. We show that increased optimism by investors is associated with relaxing the optimal leverage cap, while increased optimism by creditors, or jointly by both investors and creditors is associated with a tighter optimal leverage cap.

We show that when belief distortions and government bailouts coexist, increased optimism by equity investors may call for a tighter optimal leverage cap too, depending on whether equity optimism is concentrated on upside or downside risk. We also show that monetary tightening can act as a useful substitute for financial regulation since increased optimism by either equity investors or creditors is associated with higher incentives to raise interest rates. Finally, we decompose the more nuanced effects of beliefs on optimal policy when leverage caps are imperfectly targeted.

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A Proofs and derivations: Section 2

We prove the results in this section, with the exception of Proposition 3, in a general model that allows for monetary policy and bailouts. Therefore, the proofs presented here also establish the corresponding results in Sections 4 and 5. Our baseline model is recovered by setting bailouts $t(b,s)=0$ and the discount factor implied by monetary policy $\beta(r)=\beta(r^\star)=\beta^C$.

Proof of Lemma 1

The problem that investors face initially, after anticipating their optimal default decision, can be expressed as follows

$$V\left(\bar{b},r\right)=\max_{b,k,c_0^I,c_1^I(s)} c_0^I+\beta^I\int c_1^I(s)\,dF^I(s),$$

subject to budget constraints at date 0 and in each state s at date 1, the creditors' debt pricing equation, the non-negativity constraint of consumption, and the regulatory leverage constraints

$$\begin{aligned}
c_0^I+k+\Upsilon(k)&=n_0^I+Q(b)\,k \quad (\lambda_0)\\
c_1^I(s)&=n_1^I(s)+\max\{s+t(b,s)-b,0\}\,,\forall s\\
Q(b,r)&=\beta(r)\left(\int_{s^\star(b)}^{\overline{s}} bdF^C(s)+\int_{\underline{s}}^{s^\star(b)}(\phi s+t(b,s))\,dF^C(s)\right)\\
c_0^I&\geq 0 \qquad (\eta_0)\\
bk&\leq \bar{b}k \quad (\mu)\,.
\end{aligned}$$

We can express this problem in Lagrangian form as follows

$$\mathcal{L}=c_0^I+\beta^I\int_{s^\star(b)}^{\overline{s}}(s+t(b,s)-b)\,dF^I(s)\,k-\lambda_0\left(c_0^I-n_0^I-Q(b,r)\,k+k+\Upsilon(k)\right) \tag{29}$$

$$+\eta_0c_0^I+\mu k\left(\bar{b}-b\right). \tag{30}$$

Equations (2) and (3), as well as the results in Lemma 5, follow directly from Equation (29) when $\eta_0=0$.

Proof of Proposition 1

Equation (6) follows from maximizing Equation (2) subject to Equation (3) in the text. Equations (7) and (8) follow from differentiating Equations (2) and (4) in the text. These conditions are necessary for optimality and sufficient under the regularity conditions described below.

Proof of Lemma 2

Differentiating Equation (8) with respect to $\bar{b}$ implies that $\frac{dM}{db}\left(b^{\star}\right)\frac{db^{\star}}{d\bar{b}} = \Upsilon''\left(k^{\star}\right)\frac{dk^{\star}}{d\bar{b}}$. Equation (9) follows immediately by rearranging this expression and noticing that $\frac{db^{\star}}{d\bar{b}} = 1$ when $\mu > 0$ and $\frac{db^{\star}}{d\bar{b}} = 0$ when $\mu = 0$.

Proof of Lemma 3

Note the first-order condition for leverage can be written as $\frac{dM}{db}\left(b^{\star}; F^I, F^C\right) = 0$. An application of the implicit function theorem implies that

$$\frac{d^2M}{db^2}\left(\frac{\delta b^{\star}}{\delta F^I}\cdot G^I\right) + \frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I}\cdot G^I = 0 \Rightarrow \frac{\delta b^{\star}}{\delta F^I}\cdot G^I = \frac{\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I}\cdot G^I}{-\frac{d^2M}{db^2}}.$$

The same approach applies when (variationally) differentiating with respect to F^C.

Similarly, the first-order condition for leverage can be written as $\frac{dM}{db}\left(b^{\star}; F^I, F^C\right) = 0$. An application of the implicit function theorem implies that

$$\frac{dM}{db}\left(\frac{\delta b^{\star}}{\delta F^I}\cdot G^I\right) + \frac{\delta M}{\delta F^I}\cdot G^I = \Upsilon''\left(k^{\star}\right)\left(\frac{\delta k^{\star}}{\delta F^I}\cdot G^I\right) \Rightarrow \frac{\delta k^{\star}}{\delta F^I}\cdot G^I = \frac{\frac{dM}{db}\left(\frac{\delta b^{\star}}{\delta F^I}\cdot G^I\right) + \frac{\delta M}{\delta F^I}\cdot G^I}{\Upsilon''\left(k^{\star}\right)} = \frac{\frac{\delta M}{\delta F^I}\cdot G^I}{\Upsilon''\left(k^{\star}\right)}.$$

Notice that this derivation exploits the fact that $\frac{dM}{db} = 0$. The same approach applies when (functionally) differentiating with respect to F^C.

Proof of Proposition 2

For completeness, we include here Equations (4) and (7):

$$M\left(b; F^I, F^C\right) = \beta^I \int_{s^{\star}}^{\bar{s}} \left(s + t\left(b, s\right) - b\right) dF^I\left(s\right) + \beta\left(r\right)\left(\int_{s^{\star}}^{\bar{s}} b dF^C + \int_{\underline{s}}^{s^{\star}} \left(\phi s + t\left(b, s\right)\right) dF^C\left(s\right)\right)$$

$$\frac{dM}{db}\left(b; F^I, F^C\right) = \beta\left(r\right)\int_{s^{\star}(b)}^{\bar{s}} dF^C\left(s\right) - \beta^I \int_{s^{\star}(b)}^{\bar{s}} dF^I\left(s\right) - \left(1-\phi\right)\beta\left(r\right) s^{\star}\left(b\right) f^C\left(s^{\star}\left(b\right)\right).$$

We compute $\frac{\delta M}{\delta F^I} \cdot G^I$ as follows:

$$
\begin{aligned}
\frac{\delta M}{\delta F^I} \cdot G^I &\equiv \lim_{\varepsilon \to 0} \frac{M\left(b; F^I + \varepsilon G^I, F^C\right) - M\left(b; F^I, F^C\right)}{\varepsilon} \\
&= \lim_{\varepsilon \to 0} \frac{\beta^I \int_{s^\star}^{\overline{s}} (s + t(b,s) - b)\, d\left(F^I(s) + \varepsilon G^I(s)\right) - \beta^I \int_{s^\star}^{\overline{s}} (s + t(b,s) - b)\, dF^I(s)}{\varepsilon} \\
&= \beta^I \int_{s^\star}^{\overline{s}} (s + t(b,s) - b)\, dG^I(s) \\
&= -\beta^I \int_{s^\star}^{\overline{s}} \left(1 + \frac{\partial t(b,s)}{\partial s}\right) G^I(s)\, ds
\end{aligned}
$$

where the last line follows after integrating by parts.

We compute $\frac{\delta M}{\delta F^C} \cdot G^C$ as follows:

$$
\begin{aligned}
\frac{\delta M}{\delta F^C} \cdot G^C &\equiv \lim_{\varepsilon \to 0} \frac{M\left(b; F^I, F^C + \varepsilon G^C\right) - M\left(b; F^I, F^C\right)}{\varepsilon} \\
&= \beta(r) \left(\int_{s^\star}^{\overline{s}} b dG^C(s) + \int_{\underline{s}}^{s^\star} (\phi s + t(b,s))\, dG^C(s) \right) \\
&= -\beta(r) \left[(1-\phi)\, s^\star G^C(s^\star) + \int_{\underline{s}}^{s^\star} \left(\phi + \frac{\partial t(b,s)}{\partial s}\right) G^C(s)\, ds \right]
\end{aligned}
$$

where the last line follows after integrating by parts.

We compute $\frac{\delta \frac{dM}{db}}{\delta F^I} \cdot G^I$ as follows:

$$
\begin{aligned}
\frac{\delta \frac{dM}{db}}{\delta F^I} \cdot G^I &= \lim_{\varepsilon \to 0} \frac{\left(-\beta^I \int_{s^\star}^{\overline{s}} d\left(F^I + \varepsilon G^I\right) + \beta^I \int_{s^\star}^{\overline{s}} \frac{\partial t}{\partial b} d\left(F^I + \varepsilon G^I\right)\right) - \left(-\beta^I \int_{s^\star}^{\overline{s}} dF^I + \beta^I \int_{s^\star}^{\overline{s}} \frac{\partial t}{\partial b} dF^I\right)}{\varepsilon} \\
&= \beta^I \left[-\int_{s^\star}^{\overline{s}} dG^I(s) + \int_{s^\star}^{\overline{s}} \frac{\partial t(b,s)}{\partial b} dG^I(s) \right] \\
&= \beta^I \left[\left(1 - \frac{\partial t(b, s^\star(b))}{\partial b}\right) G^I(s^\star(b)) - \int_{s^\star(b)}^{\overline{s}} \frac{\partial^2 t(b,s)}{\partial b \partial s} G^I(s)\, ds \right]
\end{aligned}
$$

We compute $\frac{\delta \frac{dM}{db}}{\delta F^C} \cdot G^C$ as follows:

$$
\begin{aligned}
\frac{\delta \frac{dM}{db}}{\delta F^C} \cdot G^C &= \beta(r) \left(\int_{s^\star}^{\overline{s}} dG^C(s) - (1-\phi)\, s^\star(b)\, g^C(s^\star) \frac{\partial s^\star}{\partial b} + \int_{\underline{s}}^{s^\star} \frac{\partial t}{\partial b} dG^C(s) \right) \\
&= -\beta(r) \left[G^C(s^\star(b)) \frac{\partial s^\star(b)}{\partial b} \left(1 + \frac{\partial t(b, s^\star(b))}{\partial s} + (1-\phi) s^\star(b) \frac{g^C(s^\star(b))}{G^C(s^\star(b))}\right) + \int_0^{s^\star(b)} \frac{\partial^2 t(b,s)}{\partial b \partial s} G^C(s)\, ds \right]
\end{aligned}
$$

where the last line follows after integrating by parts and using the fact that $\frac{\partial s^\star}{\partial b} = 1$.

Proof of Proposition 3

a) From Lemma 3, it follows that $\frac{\delta b^\star}{\delta F^I} \cdot G^I$ and $\frac{\delta k^\star}{\delta F^I} \cdot G^I$ will have the same sign as $\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I} \cdot G^I$ and $\frac{\delta M}{\delta F^I} \cdot G^I$, respectively. From Equations (10) and (12), if investors are optimistic in a hazard rate sense, $G^I(\cdot) \leq 0$, and it is immediate that $\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I} \cdot G^I < 0$ and $\frac{\delta M}{\delta F^I} \cdot G^I > 0$, and therefore $\frac{\delta b^\star}{\delta F^I} \cdot G^I < 0$ and $\frac{\delta k^\star}{\delta F^I} \cdot G^I > 0$.

b) From Lemma 3, it follows that $\frac{\delta b^\star}{\delta F^C} \cdot G^C$ and $\frac{\delta k^\star}{\delta F^C} \cdot G^C$ will have the same sign as $\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^C} \cdot G^C$ and $\frac{\delta M}{\delta F^C} \cdot G^C$, respectively. From Equations (11) and (13), if creditors are optimistic in a hazard rate sense, $G^C(\cdot) \leq 0$ and $1 - (1-\phi)\, s \frac{g^C(s)}{G^C(s)} > 0$, and it is immediate that $\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^C} \cdot G^C > 0$ and $\frac{\delta M}{\delta F^C} \cdot G^C > 0$, and therefore $\frac{\delta b^\star}{\delta F^C} \cdot G^C > 0$ and $\frac{\delta k^\star}{\delta F^I} \cdot G^I > 0$.

c) Suppose that $F^C = F^I = F^0$. Then the effect of joint exuberance on $\frac{dM}{db}$ is

$$\begin{aligned}
\frac{\delta\left(\frac{dM}{db}\right)}{\delta F^0} \cdot G &= \frac{\delta\left(\frac{dM}{db}\right)}{\delta F^I} \cdot G + \frac{\delta\left(\frac{dM}{db}\right)}{\delta F^C} \cdot G \\
&= -G\left(s^\star(b)\right)\left(\beta(r) - \beta^I + \beta(r)(1-\phi)\, s^\star(b) \frac{g\left(s^\star(b)\right)}{G\left(s^\star(b)\right)}\right).
\end{aligned}$$

Since optimism in a hazard rate sense implies that $G\left(s^\star(b)\right) \leq 0$, we need to show that

$$\beta(r) - \beta^I + \beta(r)(1-\phi)\, s^\star(b) \frac{g\left(s^\star(b)\right)}{G\left(s^\star(b)\right)} \geq 0.$$

At an interior optimum with common beliefs, Equation (7) implies that

$$\frac{dM}{db} = \beta(r) - \beta^I - \beta(r)(1-\phi)\, s^\star(b) \frac{f^0\left(s^\star(b)\right)}{1 - F^0\left(s^\star(b)\right)} = \mu \geq 0$$

or, equivalently,

$$\beta(r) - \beta^I \geq \beta(r)(1-\phi)\, s^\star(b) \frac{f^0\left(s^\star(b)\right)}{1 - F^0\left(s^\star(b)\right)}.$$

As shown in Section E.3, hazard rate dominance implies that $\frac{f^0(s)}{1-F^0(s)} \geq -\frac{g^0(s)}{G^0(s)}$, so we get

$$\beta(r) - \beta^I \geq -\beta(r)(1-\phi)\, s^\star(b) \frac{g^0(s)}{G^0(s)},$$

which implies, as required, that

$$\beta(r) - \beta^I + \beta(r)(1-\phi)\, s^\star(b) \frac{g^0(s)}{G^0(s)} \geq 0.$$

B Proofs and derivations: Section 3

We prove the results in this section in a general model that allows for monetary policy and bailouts. Therefore, the proofs presented here also establish the corresponding results in Sections 4 and 5. Our baseline model is recovered by setting bailouts $t(b,s)=0$ and the discount factor implied by monetary policy $\beta(r)=\beta(r^\star)=\beta^C$.

Proof of Lemma 4

The planner's objective is given by the sum of investors' and creditors' expected utility. Formally, ignoring constant terms that depend only on endowments, we have $W=u^{I,P}+u^{C,P}$, where $u^{I,P}$ and $u^{C,P}$ are given by

$$
\begin{aligned}
u^{I,P} &= k\left[Q(b,r)-1+\beta^I\int_{s^\star(b)}^{\overline{s}}(s+t(b,s)-b)\,dF^{I,P}(s)\right]-\Upsilon(k)\\
u^{C,P} &= k\left[-Q(b,r)+\beta^C\left(\int_{s^\star(b)}^{\overline{s}}bdF^{C,P}+\int_{\underline{s}}^{s^\star(b)}(\phi s+t(b,s))\,dF^{C,P}(s)\right)\right],
\end{aligned}
$$

which imply that

$$W=\left[\beta^I\int_{s^\star(b)}^{\overline{s}}(s+t(b,s)-b)\,dF^{I,P}(s)+\beta^C\left(\int_{s^\star(b)}^{\overline{s}}bdF^{C,P}+\int_{\underline{s}}^{s^\star(b)}(\phi s+t(b,s))\,dF^{C,P}(s)\right)-1\right]k-\Upsilon(k$$

We define $M^P(b)$ as follows:

$$M^P(b)=\beta^I\int_{s^\star(b)}^{\overline{s}}(s+t(b,s)-b)\,dF^{I,P}(s)+\beta^C\left(\int_{s^\star(b)}^{\overline{s}}bdF^{C,P}+\int_{\underline{s}}^{s^\star(b)}(\phi s+t(b,s))\,dF^{C,P}(s)\right).$$

The results in Lemmas 4 and 6 follow immediately by setting $t(b,s)=0$.

Proof of Proposition 4

The result follows directly by totally differentiating our characterization of welfare in Lemma 4, applying the envelope theorem, and noting that $\frac{db^\star}{d\bar{b}}=1$ whenever the leverage cap is binding. Its general version with bailouts and monetary policy is

$$\frac{dW}{d\bar{b}}=\underbrace{\left[\frac{dM^P\left(\bar{b}\right)}{d\bar{b}}-\frac{d\gamma(b)}{db}\right]k^\star\left(\bar{b},r\right)}_{\text{Inframarginal Effect}}+\underbrace{\left[M^P\left(\bar{b}\right)-\gamma(b)-M\left(\bar{b}\right)\right]\frac{dk^\star\left(\bar{b},r\right)}{d\bar{b}}}_{\text{Incentive Effect}}.$$

Proof of Proposition 5

The variational derivative of Equation (14) with respect to beliefs F^j for $j \in \{I, C\}$ is

$$\begin{aligned}\frac{\delta \frac{dW}{db}}{\delta F^j} \cdot G^j = & \left[\frac{dM^P\left(\bar{b}\right)}{db} - \frac{d\gamma\left(b\right)}{db}\right]\left[\frac{\delta k\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right] \\ & + \left[M^P\left(\bar{b}\right) - \gamma\left(b\right) - M\left(\bar{b}\right)\right]\left[\frac{\delta \frac{dk}{db}}{\delta F^j} \cdot G^j\right] - \left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right]\frac{dk\left(\bar{b}\right)}{d\bar{b}}\end{aligned}$$

Notice that

$$k\left(\bar{b}\right) = \varphi\left(M\left(\bar{b}\right) - 1\right)$$

which implies

$$\frac{\delta k\left(\bar{b}\right)}{\delta F^j} \cdot G^j = \varphi'\left(\cdot\right)\left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right]$$

and

$$\frac{dk\left(\bar{b}\right)}{d\bar{b}} = \varphi'\left(\cdot\right)\frac{dM\left(\bar{b}\right)}{db}$$

Hence, the last term in the variational derivative is

$$\begin{aligned}\left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right]\frac{dk\left(\bar{b}\right)}{d\bar{b}} &= \frac{dM\left(\bar{b}\right)}{db}\varphi'\left(.\right)\left[\frac{\delta M\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right] \\ &= \frac{dM\left(\bar{b}\right)}{db}\left[\frac{\delta k\left(\bar{b}\right)}{\delta F^j} \cdot G^j\right].\end{aligned}$$

Combining, we obtain the required expression in Equation (16), and also the result in Proposition 4 with bailouts.

Proof of Proposition 6

The results in this proposition follow directly by combining the comparative statics in Propositions 3 with the general characterization in Proposition 5.

C Proofs and derivations: Sections 4 and 5

In Appendices A and B, we have proved the results in sections 2 and 3 using a general model that allows for flexible monetary policy and bailouts. Therefore, the results in sections 4 and 5 follow immediately from those characterizations.

D Proofs and derivations: Section 6

Proof of Proposition 11

Let average beliefs be $F^j(s) = \int_{\underline{\xi}}^{\bar{\xi}} F^j(s;\xi)\, dH(\xi)$, $j \in \{I, H\}$, and let the market valuation under average beliefs be $M(b)$, as defined in Equation (4). Since $M(b)$ and $\frac{dM(b)}{db}$ linear functionals of beliefs, we have $M(b) = E\left[M\left(\bar{b};\xi\right)\right]$ and $\frac{dM(b)}{db} = E\left[\frac{dM\left(\bar{b};\xi\right)}{db}\right]$. Now consider he value of $\frac{dW}{db}$ in the benchmark with perfect targeting, which we denote as $\frac{dW^0}{db}$. Using quadratic costs $\Upsilon(k) = \frac{1}{2\varphi}k^2$ in Proposition 4, we get

$$\frac{dW^0}{d\bar{b}} = \varphi\left(\frac{dM^P\left(\bar{b}\right)}{d\bar{b}}E\left[M\left(\bar{b};\xi\right)-1\right] + E\left[M^P\left(\bar{b};\xi\right) - M\left(\bar{b};\xi\right)\right]E\left[M\left(\bar{b};\xi\right)\right]\right)$$

Moreover, with quadratic costs, Equation (28) with imperfect targeting becomes

$$E\left[\frac{dW}{d\bar{b}}\right] = \int\limits_{\{\xi:\mu(\xi)>0\}} \frac{d\hat{W}\left(\bar{b};\xi\right)}{d\bar{b}}dH(\xi)$$

where

$$\frac{d\hat{W}\left(\bar{b};\xi\right)}{d\bar{b}} \equiv \varphi\left[\frac{dM^p\left(\bar{b}\right)}{db}\left(M\left(\bar{b};\xi\right)-1\right) + \left(M^p\left(\bar{b}\right) - M\left(\bar{b};\xi\right)\right)\frac{dM\left(\bar{b};\xi\right)}{d\bar{b}}\right]$$

Moreover, we can write

$$\int_{\underline{\xi}}^{\bar{\xi}} \frac{d\hat{W}\left(\bar{b};\xi\right)}{d\bar{b}}dH(\xi) = \frac{dW^0}{d\bar{b}} - \varphi Cov\left[M\left(\bar{b};\xi\right), \frac{dM\left(\bar{b};\xi\right)}{d\bar{b}}\right]$$

so that we obtain

$$E\left[\frac{dW}{d\bar{b}}\right] = \frac{dW^0}{d\bar{b}} - \varphi\cdot Cov\left[M\left(\bar{b};\xi\right), \frac{dM\left(\bar{b};\xi\right)}{d\bar{b}}\right] - \int\limits_{\{\xi:\mu(\xi)\leq 0\}} \frac{d\hat{W}\left(\bar{b};\xi\right)}{db}dH(\xi)$$

as required.

Proof of Proposition 11

This result follows directly by noting that the third term in (28) is zero when the constraint is binding for all ξ, and evaluating the covariance terms in each scenario using the comparative statics in Proposition 3.

Online Appendix

E Additional proofs and derivations

E.1 Regularity conditions

Note that investors always find optimal to choose non-negative leverage in equilibrium, since

$$\left.\frac{dM}{db}\right|_{b=0} = \beta^C - \beta^I \geq 0.$$

Therefore, for a given leverage constraint $\bar{b}$, our problem always features a solution for leverage in $\left[0, \bar{b}\right]$ and a finite solution for investment, since $\frac{d^2V}{dk^2} = -\Upsilon''(k) < 0$. To guarantee a finite solution under laissez-faire, we simply need that creditors perceive the net prevent value of investment to be negative if there is always default, that is, $\beta^C \phi \mathbb{E}^C[s] < 1$, since

$$\lim_{b\to\infty} M(b) = \beta^C \phi \mathbb{E}^C[s].$$

These results extend directly to the environment studied in Sections 4 and 5 after imposing that bailouts are bounded above, $t(b,s) \leq \bar{t}$, and that investment has negative NPV if always in distress, even with a big bailout, $\beta(r)\left(\phi \mathbb{E}^C[s] + \bar{t}\right) < 1$.

In order to explore the second-order condition on investors' leverage choice, it is useful to normalize $\frac{dM}{db}$, characterized in Equation (7), as follows

$$J(b) = \frac{\frac{dM}{db}}{\beta^C\left(1 - F^C(b)\right)} \equiv 1 - \frac{\beta^I}{\beta^C}\frac{1 - F^I(b)}{1 - F^C(b)} - (1-\phi)\, b \frac{f^C(b)}{1 - F^C(b)}.$$

Therefore, it follows that the quasi-concavity of the investor's objective can be established by characterizing the conditions under which $J'(b)$ is negative. Note that (dividing is valid for any interior b)

$$J'(b) = -\frac{\beta^I}{\beta^C}\frac{\partial}{\partial b}\left(\frac{1 - F^I(b)}{1 - F^C(b)}\right) - (1-\phi)\left[\frac{f^C(b)}{1 - F^C(b)} + b\frac{\partial}{\partial b}\left(\frac{f^C(b)}{1 - F^C(b)}\right)\right]$$

There are two sufficient conditions that guarantee that $J'(b) < 0$. First, monotone hazard rates imply that

$$\frac{\partial}{\partial b}\left(\frac{f^C(b)}{1 - F^C(b)}\right) > 0,$$

and if investors are more optimistic than creditors in the hazard rate sense, we get

$$\frac{\partial}{\partial b}\left(\frac{1-F^{I}(b)}{1-F^{C}(b)}\right)>0.$$

Hazard rate dominance is equivalent to saying that $\frac{1-F^{I}(b)}{1-F^{C}(b)}$ is increasing on b. Therefore, combining both, we have $J'(b)<0$, which gives the result. We formally state this result as Lemma 8.

Lemma 8. ***(Single-peaked objective function without bailouts)*** *Suppose that there is no bailout, and:*

1. *Shareholders are weakly more optimistic than creditors in the hazard rate order*
2. *Creditors' hazard rate* $\frac{f^{C}(s)}{1-F^{C}(s)}$ *is increasing in* s

Then $M(b)$ *is single peaked.*

Notice that the solution for optimal leverage can be expressed in general as follows

$$b=\frac{1}{(1-\phi)\frac{f^{C}(b)}{1-F^{C}(b)}}\left(1-\frac{\beta^{I}}{\beta^{C}}\frac{1-F^{I}(b)}{1-F^{C}(b)}\right).$$

E.2 First-best corrective policy

The first-best problem when the planner can control both b and k is

$$\max_{b,k} W(b,k)=\left[M^{P}(b)-1\right]k-\Upsilon(k)$$

with first-order conditions

$$\frac{dM^{P}}{db}\left(b^{1}\right)=0$$
$$M^{P}\left(b^{1}\right)-1=\Upsilon'\left(k^{1}\right),$$

where we denote by b^1 and k^1 the first-best leverage and investment. Consider an equilibrium with Pigovian taxes $\tau=(\tau_k,\tau_b)$, where investors pay $\tau_k k+\tau_b b$ at date 0 to the government, which is then rebated as a lump sum to either investors or creditors. Investors solve

$$V(\tau)=\max_{b,k}\left[M(b)-1\right]k-\Upsilon(k)-\tau_k k-\tau_b b,$$

with first-order condition

$$\frac{dM}{db}(b)\,k = \tau_b$$
$$M(b) - 1 = \Upsilon'(k) + \tau_k.$$

It follows that the Pigovian takes that achieve the first-best solution are

$$\tau_b = \left[\frac{dM}{db}\left(b^1\right) - \frac{dM^P}{db}\left(b^1\right)\right]k^1$$
$$\tau_k = M\left(b^1\right) - M^P\left(b^1\right)$$

E.3 Properties of hazard-rate dominant perturbations

The hazard rate after an arbitrary perturbation of the form described in Section 2 of the paper is given by $h(s) = \frac{f(s)+\varepsilon g(s)}{1-(F(s)+\varepsilon G(s))}$. Its derivative with respect to ε takes the form

$$\frac{dh(s)}{d\varepsilon} = \frac{g(s)}{1-(F(s)+\varepsilon G(s))} + \frac{(f(s)+\varepsilon g(s))\,G(s)}{(1-(F(s)+\varepsilon G(s)))^2}.$$

When we take the limit $\varepsilon \to 0$, hazard rate dominance means that $\lim_{\varepsilon\to 0}\frac{dh(s)}{d\varepsilon} \leq 0$, therefore

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{dh(s)}{d\varepsilon} &= \frac{g(s)}{1-F(s)} + \frac{f(s)}{1-F(s)}\frac{G(s)}{1-F(s)} \leq 0 \\
&\Leftrightarrow g(s) + \frac{f(s)}{1-F(s)}G(s) \leq 0 \\
&\Leftrightarrow \frac{g(s)}{G(s)} + \frac{f(s)}{1-F(s)} \geq 0 \\
&\Leftrightarrow \frac{f(s)}{1-F(s)} \geq -\frac{g(s)}{G(s)},
\end{aligned}$$

where in the second-to-last line the sign of the inequality flips because $G(s) \leq 0$.