Dissecting Fire Sales Externalities

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Abstract

This paper shows that two *distinct* types of fire sales/pecuniary externalities arise in Walrasian models with incomplete markets and/or credit (collateral) constraints. I respectively brand them terms-of-trade and collateral externalities. While terms-of-trade externalities may create over- or under-investment, collateral externalities always create over-investment. This paper also shows that ex-post constrained Pareto improvements require the use of ex-ante transfers — without transfers, no improvements are feasible in plausible environments. It also shows that the presence of amplification mechanisms is neither necessary nor sufficient to determine the efficiency of economies with credit constraints: the implications of credit constraints for amplification and welfare must be studied separately.

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1 Introduction

In recent history, modern economies have experienced recurrent boom-bust episodes involving sharp changes in asset prices, often intertwined with credit frictions. Understanding whether these episodes are associated with some form of inefficiency is, without doubt, an important question. More precisely, it is important to understand the exact mechanism by which price changes in economies with financial frictions provide a rationale for policy intervention, since we know that such rationale does not exist in frictionless economies.

In particular, fire sales/pecuniary externalities have taken a prominent role in shaping the academic agenda and many policy discussions on regulatory issues in the aftermath of the 2007/8 financial crisis.[1] The speech by Stein (2013) is a recent example of current policy thinking based on academic insights. However, despite its importance for policy design, the precise requirements for when and how fire sales/pecuniary externalities create inefficiencies remains blurred. This partly occurs because even the simplest models in which inefficiencies arise — as the one studied in this paper — require a non-trivial number of ingredients.

To shed light on the efficiency properties of environments with credit frictions, this paper studies a stylized and abstract model — a simple Walrasian borrower-lender problem — that delivers insights applicable to any environment in which credit constraints are present. For instance, the natural holders of assets in this paper (experts) can represent firms, as in He and Kondor (2014), financial intermediaries, as in Lorenzoni (2008), or financially constrained arbitrageurs, as in Gromb and Vayanos (2002).

The main contribution of this paper is to show that two *distinct* externalities associated with price changes arise in Walrasian models with incomplete markets and/or credit (collateral) constraints.

I refer to the first externality as the **terms-of-trade externality**.[2] This externality is present

[1]Whenever the owners of an asset — perhaps its natural owners, in the words of Shleifer and Vishny (1992) — happen to sell it to low valuation users, who provide a downward sloping demand, we expect to observe low equilibrium asset prices. This is the classic notion of a fire sale, which can also involve a negative spiral in which low equilibrium prices increase the pressure to sell, which further lowers equilibrium prices, and so on.

[2]Previous versions of this paper referred to this externality as the "risk-sharing externality" or the "net worth externality". The former term can be misleading because this externality is also present in environments without risk, only through intertemporal substitution effects — as in the baseline model of this paper. The latter term is also misleading because, as shown in the paper, what matters for welfare is not the level of net worth (a stock variable), but the change in net worth derived from asset sales (the flow loss/gain incurred when selling or buying an asset). Note that this paper focuses on terms-of-trade externalities regarding financial assets, but similar externalities arise through changes in the terms-of-trade of non-durable goods, durable goods or labor services as long as markets are incomplete. For instance, the externality studied in J. Davila, J. Hong, P. Krusell and J.V. Ríos-Rull (2012) is a terms-of-trade externality that works through wage changes.

as long as marginal rates of substitution (MRS) across periods/states differ across agents. MRS may not be equalized for different reasons: for instance, there may exist binding credit constraints (which may or may not depend on endogenously determined asset prices) or the set of traded assets may not span all possible states of nature (e.g., agents can only trade a noncontingent bond even though there two or more states of nature). This paper studies both situations. Intuitively, when MRS are not equal, a planner can modify allocations to induce price changes that improve the terms-of-trade of the transactions of those agents with relatively higher marginal utility in a given period/state. In particular, in a fire sale, sellers do not internalize that extra units of fire-sold capital worsen the terms-of-trade (since they receive a lower price) of other sellers of capital, who may greatly value having resources in those states.

I refer to the second externality as the **collateral externality.** This externality arises when credit constraints that depend on endogenously determined asset prices are binding. Intuitively, agents do not internalize that their decisions directly affect collateral values through price changes, changing the effective borrowing capacity of other credit constrained agents. This externality works through the shadow value that the holder of an asset attaches to borrowing against that asset, but it is unrelated to the terms-of-trade obtained when selling an asset. This externality can occur even when MRS are equal across agents for welfare purposes; for instance, in models that allow for ex-post insurance, such as those that adopt the large-family/representative agent approach. As it is clear from the analysis in section 5.2, any externality that operates by tightening a price dependent binding credit constraint, e.g., a moral hazard incentive constraint or a value-at-risk requirement, is of the same nature as the collateral externality. In those cases, it may be more appropriate to use the term margin externality or binding-price-constraint externality instead of collateral externality.

Although both terms-of-trade and collateral externalities already exist in isolation in the literature, there has been confusion on how they relate to each other. This is the first paper that studies both externalities at the same time, which allows us to isolate how both externalities differentially affect welfare. The contribution of this paper is not only to present both types of externalities in the same model but, more importantly, to acknowledge that two completely different mechanisms are at play. This distinction is not present in the existing literature: a large number of papers that feature only terms-of-trade externalities incorrectly refer to papers that only feature collateral externalities as if they shared the same economic mechanism and viceversa.

This paper shows that the terms-of-trade externality may cause over- or under-investment, while the collateral externality always creates over-investment. Hence, the direction of the policy intervention when terms-of-trade externalities are present crucially relies on the structure of the economy, unlike in the collateral externality case. This paper also derives several novel insights for practical policy design that can only be drawn when differentiating both types of externalities in a unified framework — see, in particular, the remarks in section 3.3. It will not be sufficient for future work to argue that a paper contains a fire sale/pecuniary externality, it will be important to identify which type of externality — either of the terms-of-trade type or the collateral type — is present.

Several additional new results emerge from the analysis. First, this paper shows that, when fire sales occur in a single period/state, ex-post constrained Pareto improvements in general require ex-ante transfers — that is, the competitive allocation would be efficient if transfers were not allowed.[3] This is an important result, since it is often argued that the generic constrained inefficiency results of Geanakoplos and Polemarchakis (1986) or Greenwald and Stiglitz (1986) always justify policy interventions, precisely because they are generic. This paper shows that, in plausible situations, economies may perfectly be constrained efficient. This paper carefully studies how the set of instruments available to the policymaker and the welfare criterion used by the policymaker do matter to assess the efficiency of economies with credit constraints.

Second, this paper also shows that the amplification generated by binding constraints and the associated welfare implications must be studied separately.[4] The paper provides examples of a constrained inefficient equilibrium without amplification and of a efficient equilibrium with amplification. This result shows that causal statements of the kind "price spirals cause externalities", which are widespread in both academic work and policy discussions, are misleading. Cooper and John (1988) forcefully make a similar argument in a different environment by distinguishing between spillovers (payoff relevant externalities) and complementarities (amplification).

Third, this paper shows that the optimal policy of a planner who seeks to implement constrained Pareto improvements is time inconsistent. Both types of externalities, precisely because of their intertemporal/across-state nature, generate time inconsistency, a result not studied in previous work.

[3]The welfare benchmark used in this paper is that of constrained Pareto efficiency, in which the planner faces the same constraints as the agents in the decentralized market. The paper analyzes the cases with and without ex-ante transfers.

[4]I use the term amplification to refer to situations in which both demand and supply curves for an asset have the same slope, which implies that small changes in fundamentals may have large effects on equilibrium outcomes. Amplification due to financial constraints may be important quantitatively and it often appears in environments in which externalities are present. This paper shows that the existence of amplification is not what causes the externalities or viceversa; this paper does not say that amplification mechanisms and externalities cannot appear simultaneously or that amplification is unimportant quantitatively.

Related Literature

This paper relates to the literature on amplification mechanisms and financial constraints in credit markets — see Krishnamurthy (2010) or Brunnermeier and Oehmke (2012) for recent surveys. Although the bulk of this literature has focused on the positive effects of financial constraints on amplification and persistence, I only discuss papers whose focus is welfare analysis based on fire sales/pecuniary externalities. Shleifer and Vishny (2011) give an overview of mostly positive results on fire sales.

The papers by Caballero and Krishnamurthy (2001), Gromb and Vayanos (2002), Lorenzoni (2008), Korinek (2009), Hart and Zingales (2011) and He and Kondor (2014) contain externalities of the kind described in this paper as **terms-of-trade externalities**. This externality can be traced back to the seminal paper by Hart (1975), who provides the first example of constrained inefficiency when markets are incomplete. Also exploiting terms-of-trade externalities, Geanakoplos and Polemarchakis (1986) show that economies with (exogenous) incomplete markets are generically ex-post constrained inefficient when the number of states/periods and/or the set of traded securities is sufficiently large in comparison to the number of agents in the economy.

The literature that studies liquidity provision and the coexistence of financial intermediaries and markets, for instance, Jacklin (1987), Bhattacharya and Gale (1987), Allen and Gale (2004) and Farhi, Golosov and Tsyvinski (2009), is also related to the terms-of-trade externality. In this line of work, the possibility of spot retrading in financial markets, together with market incompleteness, reduces risk sharing opportunities, making regulation desirable. In these situations, redistributing resources through price changes between different types of agents may be welfare improving in an ex-ante sense, i.e., across types of agents.

The papers by Bianchi (2011), Bianchi and Mendoza (2012), Jeanne and Korinek (2010), Kilenthong and Townsend (2010), Stein (2012), Gersbach and Rochet (2013) and Benigno et al. (2013) contain a different type of externality, described in this paper as a **collateral externality**.[5] By using a representative agent framework to calculate welfare, these papers shut down the mechanism that creates the terms-of-trade externality. This growing literature argues that their results fall under the umbrella of the Geanakoplos and Polemarchakis (1986) generic inefficiency result, but the present paper shows that the mechanism behind collateral externalities is a different one.[6] Showing that collateral externalities are unrelated to the generic

[5] The externality in Gersbach and Rochet (2013) arises through a price-dependent credit constraint, not a collateral constraint, so it may be more appropriate to refer to it as a margin externality or binding-price-constraint externality. See section 5.2. The same margin/binding-price-constraint externality would arise when studying efficiency in a model like Brunnermeier and Pedersen (2009).

[6] Although Greenwald and Stiglitz (1986) do not analyze the case of collateral constraints directly, the collateral

inefficiency result in Geanakoplos and Polemarchakis (1986), which is often used to advocate for regulatory interventions, is an important corollary of this paper.

Outline Section 2 lays out the baseline model and analyzes the competitive equilibrium. Section 3 conducts the normative analysis, identifying the different externalities and characterizing the efficiency of the economy under different welfare benchmarks. Section 4 shows the dichotomy between amplification and welfare and studies the possibility of time inconsistency. Section 5 analyzes several extensions of the baseline model and section 6 concludes.

2 Baseline model

This section introduces the baseline model, which presents the main results in a stylized environment. Section 5 relaxes several assumptions.

2.1 Environment

Time is discrete and there are three dates $t = 0, 1, 2$. There is no uncertainty. There is a unit measure of experts (e) and a unit measure of financiers (f). Experts represent the natural holders of capital (high valuation users), who may be subject to credit constraints. Financiers represent unsophisticated holders of capital (low valuation users), who take unconstrained financial positions. The fire sale of capital occurs in the intermediate period $t = 1$.

Experts There are two goods in this economy: a consumption good (dollar), which acts as numeraire, and a capital good. Experts' utility is linear in the consumption good and given by:

$$U^e = \xi_0 c_0 + \xi_1 c_1 + \xi_2 c_2,$$

where all ξ_t are positive scalars. Varying the values of ξ_t allows to parametrize investment opportunities/preferences for experts.[7] Experts' consumption cannot be negative, that is,

externality can be interpreted as a special case within their general framework. Because investment opportunities for the agents in the economy depend directly on prices (not only through budget constraints), and those agents do not internalize the fact that their decisions affect prices, the competitive allocation will not in general be Pareto optimal: prices play the dual role of signaling scarcity and determining access to credit.

[7]The preferred interpretation for the parameters ξ_t is that they capture investment opportunities for experts. For instance, experts may have an exclusive (within period) technology that yields ξ_t units of consumption good per unit of input. Alternatively, they could be interpreted as purely preference parameters, capturing different preferences for intertemporal consumption, or as a reduced form representation of corporate risk management concerns, as in Froot, Scharfstein and Stein (1993) or Rampini and Viswanathan (2010).

$c_t > 0$. To avoid clutter, variables regarding experts have no superscripts.

Experts have endowments w_0, w_1 and w_2 of the consumption good. At $t = 0$, experts can transform units of consumption good into capital at rate p_0, that is, they have access to a linear technology, which pins down the effective price of capital p_0. Experts choose the amount of capital k_0 to hold at $t = 0$. This capital does not depreciate and yields $G_1(k_0)$ units of consumption good at $t = 1$, where $G_1(\cdot)$ is an increasing and concave production technology that satisfies Inada conditions, that is, $\lim_{k_0 \to 0} G_1'(k_0) = \infty$. Likewise, at $t = 1$, experts choose the amount of capital k_1 to hold, which yields $G_2(k_1)$ units of consumption good at $t = 2$. $G_2(\cdot)$ is another increasing and concave production technology that satisfies Inada conditions. Experts sell (in equilibrium) $k_0 - k_1$ units to financiers at $t = 1$. Experts take as given the price of capital at $t = 1$, which is denoted by p_1 and determined in equilibrium.

Experts can borrow (if $a_t < 0$) or save (if $a_t > 0$) from financiers at $t = 0$ and $t = 1$ at a rate R, which is determined in equilibrium. Experts face the following credit constraints:

$$-\phi_t p_t k_t \leq a_t \quad \text{and} \quad a_t \leq S_t,\ t = 0, 1$$

where $\phi_t \in [0, 1)$ and $S_t \in [0, \infty]$. These credit constraints limit the amount that experts can borrow or save. Limited commitment on the side of financiers motivates the constraints on saving — if financiers fail to repay, experts can only recover S_t dollars. Limited commitment on the side of experts motivates the constraints on borrowing — financiers can only recover a fraction ϕ_t of the dollar value of capital $p_t k_t$, since experts can run away, at period t, with the remaining $1 - \phi_t$ fraction.[8] Experts do not have the option of defaulting at $t = 1$.

Financiers Financiers' utility is also linear in consumption and given by:

$$U^f = c_0^f + c_1^f + c_2^f$$

Financiers have large endowments w_t^f of the consumption good in each period so that their consumption is strictly positive. This assumption pins down equilibrium rates of return. Financiers are indifferent about consuming at different periods.

At period $t = 1$, financiers can transform capital goods into consumption goods using a decreasing return to scale production technology $F\left(k_1^f\right)$, which takes capital as input; $F(\cdot)$ is thus increasing and concave. $F(\cdot)$ also satisfies Inada conditions, which guarantees that experts are sellers of capital in equilibrium. Financiers do not have access to this technology at periods $t = 0$ and $t = 2$. Financiers can purchase capital from experts (given the assumptions on

[8] An alternative assumption on the timing of the limited commitment problem can imply that borrowing constraints depend on future prices p_{t+1}, rather than on current prices p_t — as in Kiyotaki and Moore (1997). The main insights of the paper remain unchanged in that case (a previous version of this paper studied that case).

primitives, this is always the case in equilibrium) at $t = 1$ at a price p_1, which is determined competitively in equilibrium. Note that, at $t = 1$, in equilibrium, $k_1^f = k_0 - k_1$.

Figure 1 shows a timeline of events.

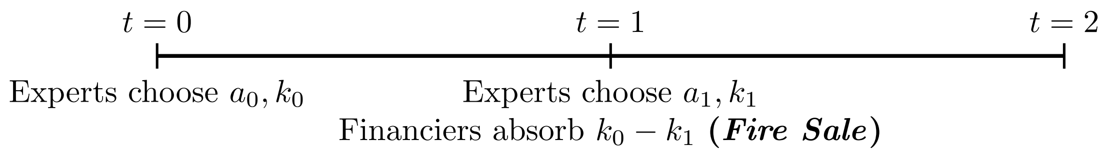

Figure 1: Timeline

From now on, I impose the following assumption, which simplifies the analysis.

Assumption 1. (Restrictions on primitives)

a) $\xi_1 > \xi_2$, which implies that experts borrow in equilibrium at $t = 1$.

b) Experts' endowments w_t are sufficiently large so that their consumption is strictly positive at all dates.

Assumption 1a) implies that experts hit their borrowing constraint at $t = 1$. It captures the notion that experts borrow as much as they can at $t = 1$ to avoid selling capital. Assumption 1b) implies that capital choices are directly determined by standard Euler equations. Consequently, it rules out the possibility of feedback loops and price spirals between low prices and binding credit constraints. I purposefully relegate the analysis with amplification to section 4 to be able to argue more forcefully that the analysis of amplification and welfare must be decoupled.

This baseline model with linear preferences is very stark. Section 5 introduces risk aversion, which allows for interior solutions, uncertainty and price-dependent margins.

2.2 Equilibrium characterization

I first solve the financiers' and experts' individual problems. Subsequently, I characterize the competitive equilibrium of the economy.

Experts' problem

The problem solved by a given expert, who behaves competitively and takes prices as given, is:

$$V^e = \max \xi_0 c_0 + \xi_1 c_1 + \xi_2 c_2$$

Subject to budget constraints:

$$c_0 + p_0 k_0 + a_0 = w_0 \qquad (\lambda_0)$$

$$c_1 + p_1 k_1 + a_1 = w_1 + G_1(k_0) + p_1 k_0 + R a_0 \qquad (\lambda_1)$$

$$c_2 = w_2 + G_2(k_1) + R a_1 \qquad (\lambda_2)$$

Credit constraints:

$$-\phi_0 p_0 k_0 \leq a_0 \ (\nu_0), \quad a_0 \leq S_0 \ (\eta_0), \qquad -\phi_1 p_1 k_1 \leq a_1 \ (\nu_1), \quad a_1 \leq S_1 \ (\eta_1)$$

And non-negativity of consumption constraints:

$$c_0 \geq 0 \ (\chi_0), \quad c_1 \geq 0 \ (\chi_1), \quad c_2 \geq 0 \ (\chi_2),$$

The Lagrange multipliers for each set of constraints, denoted by λ_t, η_t, ν_t and χ_t are defined to be (weakly) positive — see the appendix for the exact formulation of the Lagrangian. V^e denotes the indirect utility for an expert.

The optimality conditions for experts, which combined with complementary slackness conditions fully characterize the solution to the experts' problem, are given by:

$$c_t : \lambda_t = \xi_t + \chi_t,\ t = 0,1,2 \qquad \text{and} \qquad a_t : \lambda_t = R\lambda_{t+1} + \nu_t - \eta_t,\ t = 0,1$$

$$k_0 : p_0 \lambda_0 = \lambda_1 \left(G_1'(k_0) + p_1 \right) + \nu_0 \phi_0 p_0 \qquad \text{and} \qquad k_1 : p_1 \lambda_1 = \lambda_2 G_2'(k_1) + \nu_1 \phi_1 p_1 \tag{1}$$

The optimality conditions can be intuitively described using variational arguments. λ_t denotes the marginal value of a dollar at period t from the perspective of an expert. When experts' consumption is strictly positive, this marginal value is given by ξ_t — this has to be the case in equilibrium under assumption 1a). If experts decided not to consume at a given a period, χ_t would measure the additional marginal value of having an extra dollar in that state.

The optimality conditions for a_t and k_t are standard Euler equations. When credit constraints do not bind and $\nu_t = \eta_t = 0$, the choice of a_t is given by the standard $1 = \frac{R\lambda_{t+1}}{\lambda_t}$. When a credit constraint binds — note that, in equilibrium, only one constraint can bind at a given date — ν_t or η_t measure the marginal value of relaxing that constraint.

The marginal cost at $t = 0$ of buying a unit of capital is given by $p_0 \lambda_0$. In equilibrium, it must be equal to the marginal benefit of $G_1'(k_0) + p_1$ units of dollars at $t = 1$, valued at the marginal value of income λ_1, plus the marginal benefit of relaxing the binding borrowing constraints, given by $\nu_0 \phi_0 p_0$.[9] An identical logic applies to the Euler equation for k_1.

[9] Given a pattern of cash flows, investors in decentralized competitive markets value more assets that relax borrowing constraints. Fostel and Geanakoplos (2008), among other papers, have recently emphasized this idea.

Financiers' problem

The problem solved by a given financier, who behaves competitively and takes prices as given, is:

$$V^f = \max \overbrace{w_0^f + a_0^f}^{c_0^f} + \overbrace{w_1^f + a_1^f - Ra_0^f + F\left(k_1^f\right) - p_1 k_1^f}^{c_1^f} + \overbrace{w_2^f - Ra_1^f}^{c_2^f}$$

This formulation is valid because financiers' consumption is always strictly positive. V^f denotes the indirect utility for a financier.

The optimality conditions that emerge from financiers' optimization are:

$$R = 1 \text{ and } p_1 = F'\left(k_1^f\right)$$

The first condition pins down the equilibrium rate of return in borrowing/saving. The second condition delivers the downward sloping demand curve for capital at $t = 1$.

Equilibrium

The definition of equilibrium is standard. A competitive equilibrium is defined as an allocation (consumption, borrowing/lending and choices of capital) and prices (R and p_1) such that both experts and financiers behave optimally, given prices, and markets clear.[10]

Lemma 1 characterizes the competitive equilibrium of this economy.

Lemma 1. (Competitive equilibrium) *The values of k_0^{CE} and k_1^{CE} that characterize the (unique) competitive equilibrium are given by:*

$$F'\left(k_0^{CE} - k_1^{CE}\right) = p_0 \frac{\xi_0 - \nu_0 \phi_0}{\xi_1} - G_1'\left(k_0^{CE}\right) \tag{2}$$

$$F'\left(k_0^{CE} - k_1^{CE}\right) = \frac{\xi_2}{\xi_1 (1 - \phi_1) + \xi_2 \phi_1} G_2'\left(k_1^{CE}\right), \tag{3}$$

where $\nu_0 = \xi_0 - \xi_1 > 0$ if $\xi_0 > \xi_1$ and $\nu_0 = 0$ if $\xi_0 < \xi_1$. In equilibrium, $\nu_1 = \xi_1 - \xi_2 > 0$. Given the values of k_0^{CE} and k_1^{CE}, the equilibrium price is given by:

$$p_1^{CE} = F'\left(k_0^{CE} - k_1^{CE}\right)$$

The appendix characterizes the rest of endogenous variables as a function of k_0^{CE} and k_1^{CE}.

Proof. See appendix. □

[10]Note that the $k_0 = k_1 + k_1^f$ is the relevant market clearing condition for capital at $t = 1$.

Solving the system of equations defined by (2) and (3) determines k_0^{CE} and k_1^{CE} as a function of exogenous parameters. Note that experts do not sell capital because they are forced to make some payments, they sell because their preferences/technology, parametrized by ξ_1 and ξ_2, make optimal for them to do so.[11] Equations (2) and (3) essentially correspond to experts' Euler equations once equilibrium prices are substituted in.

Depending on the values of ξ_0 and ξ_1, three different cases may arise in equilibrium. Every case has different implications for the normative analysis.[12]

Case 1: *(Experts hit the saving limit at $t = 0$)* $\xi_0 < \xi_1$

In this case, investment opportunities for experts are relatively more attractive at $t = 1$ than at $t = 0$, so they decide to save resources in the initial period. Therefore $\nu_0 = 0$ and $\eta_0 = \xi_1 - \xi_0 > 0$.

Case 2: *(Experts hit the borrowing limit at $t = 0$)* $\xi_0 > \xi_1$

In this case, investment opportunities for experts are relatively more attractive at $t = 0$ than at $t = 1$, so they decide to borrow at the initial period. Therefore $\nu_0 = \xi_0 - \xi_1 > 0$ and $\eta_0 = 0$.

Case 3: *(Experts are unconstrained at $t = 0$)* $\xi_0 = \xi_1$

In this case, experts' investment opportunities are identical at $t = 0$ versus $t = 1$, so they are indifferent between borrowing and saving. Note that $\nu_0 = \eta_0 = \xi_1 - \xi_0 = 0$.

When production functions are logarithmic, the model can be solved in closed form, providing further intuition, as described in the following example.

Example. (Logarithmic production) Let's further assume that $G_1(\cdot) = G_2(\cdot) = F(\cdot) = \log(\cdot)$. In that case, we can solve in closed form for equilibrium allocation and prices. We can express the magnitude of the fire sale $k_0 - k_1$ as a proportion of k_1 as:[13]

$$\frac{k_0^{CE} - k_1^{CE}}{k_1^{CE}} = \phi_1 + \underbrace{\frac{\xi_1}{\xi_2}}_{>1} (1 - \phi_1) \tag{4}$$

[11] In section 4, experts are forced to sell capital to raise a given dollar amount, which might be a more natural form of modeling a fire sale. All welfare implications remain unchanged in that case.

[12] It is actually unclear which of these parameter combinations are likely to be more relevant in practice. On the one hand, the main result in Lorenzoni (2008) corresponds to case 1, once uncertainty is introduced — see section 5. On the other hand, within the environment of Rampini and Viswanathan (2010), which is closely related to the one used in this paper, Rampini, Sufi and Viswanathan (2014) argue that investors may decide not to hedge cash flow risk, which corresponds to the results implied by case 2 — again, once we allow for uncertainty.

[13] See the appendix for the closed form expression of k_1^{CE} as a function of primitives.

Hence, because $\phi_1 \in [0,1)$, the amount of capital sold in this economy is increasing in the difference in the marginal utility of income for experts between $t = 1$ and $t = 2$. Intuitively, when ξ_1 is relatively high, selling more units of capital at $t = 1$ is more valuable for experts.

3 Normative analysis

After characterizing the competitive equilibrium, I analyze its efficiency properties.

Welfare benchmark

The main contribution of this paper resides in the normative analysis of the equilibrium. Unfortunately, the choice of a welfare benchmark for economies with credit constraints and heterogeneous agents is not straightforward. This paper studies ex-post constrained Pareto improvements. Constrained Pareto efficiency restricts the planner to face the same choices as the agents in the decentralized market. Studying ex-post Pareto efficiency means that both groups of agents, experts and financiers, must be better off. An alternative, less restrictive, notion is that of ex-ante Pareto efficiency, in which, arguing that it is equally likely for a given agent to become and expert or a financier, an allocation is deemed efficient if it maximizes the sum of experts and financiers utilities. Initially, I assume that the planner can use ex-ante lump-sum transfers. Subsequently, I show how the results change when the planner cannot use ex-ante transfers.[14]

Previous work on related topics has used different approaches: Geanakoplos and Polemarchakis (1986) identify ex-post constrained Pareto improvements without transfers — this is the most demanding welfare criterion, which I use in section 3.2. Lorenzoni (2008) and Korinek (2009) identify ex-post constrained Pareto improvements with transfers, as I do in section 3.1. Hart and Zingales (2011) and He and Kondor (2014) identify ex-ante improvements.[15] All criteria are reasonable and equally valid; it is nonetheless important to be precise about which criterion is used because, as I show in this paper, the results regarding efficiency will in general differ.

This paper focuses on characterizing constrained efficient allocations, briefly discussing the issue of decentralization.

[14]Distinguishing whether ex-ante transfers are allowed or not is conceptually important. Given that credit constraints prevent agents from transferring cash flows across periods/states, a planner with access to ex-ante transfers has an additional instrument with respect to the agents in the decentralized economy. If the planner had access to transfers in every state/period in addition to ex-ante transfers, he could always achieve the first-best, by trivially getting around all credit constraints and replicating the complete markets allocation.

[15]In these papers, agents are ex-ante identical but they experience different (unhedgeable) shocks.

3.1 Ex-ante transfers allowed

I first analyze the case in which the planner can transfer resources between experts and financiers at $t = 0$. Specifically, I assume that the planner can transfer T_0 units of the consumption good from experts to financiers at $t = 0$. Hence, the $t = 0$ new budget constraints for borrowers and lenders respectively become:[16]

$$c_0 + p_0 k_0 + a_0 = w_0 - T_0 \qquad \text{and} \qquad c_0^f = w_0^f + a_0^f + T_0$$

Proposition 1 states the first main result of this paper.

Proposition 1. (Planner's solution with ex-ante transfers)

a) When the planner has access to ex-ante transfers, there may exist constrained Pareto improvements. The optimal choices of k_0^P and k_1^P for a planner with access to ex-ante transfers are determined by:

$$k_0 : p_0 = \frac{\lambda_1}{\lambda_0}\left(G_1'\left(k_0^P\right) + F'\left(k_0^P - k_1^P\right)\right) + \frac{\nu_0}{\lambda_0}\phi_0 p_0 + \overbrace{\underbrace{\left(\frac{\lambda_1}{\lambda_0} - 1\right)\left(k_0^P - k_1^P\right)F''\left(k_0^P - k_1^P\right)}_{\text{Terms-of-trade}} + \underbrace{\frac{\nu_1}{\lambda_0}\phi_1 k_1^P F''\left(k_0^P - k_1^P\right)}_{\text{Collateral}}}^{\text{Externality}} \tag{5}$$

$$k_1 : F'\left(k_0^P - k_1^P\right) = \frac{\lambda_2}{\lambda_1}G_2'\left(k_1^P\right) + \frac{\nu_1}{\lambda_1}\phi_1 F'\left(k_0^P - k_1^P\right) - \overbrace{\left[\underbrace{\left(1 - \frac{\lambda_0}{\lambda_1}\right)\left(k_0^P - k_1^P\right)F''\left(k_0^P - k_1^P\right)}_{\text{Terms-of-trade}} + \underbrace{\frac{\nu_1}{\lambda_1}\phi_1 k_1^P F''\left(k_0^P - k_1^P\right)}_{\text{Collateral}}\right]}^{\text{Externality}} \tag{6}$$

b) Depending on the values of ξ_1 and ξ_0, the terms-of-trade externality can have different signs:

- Case 1: If $\xi_0 < \xi_1$, the terms-of-trade externality creates over-investment ex-ante ($k_0^{CE} > k_0^P$) and over-selling ex-post ($k_1^{CE} < k_1^P$).

- Case 2: If $\xi_0 > \xi_1$, the terms-of-trade externality creates under-investment ex-ante ($k_0^{CE} < k_0^P$) and under-selling ex-post ($k_1^{CE} > k_1^P$).

- Case 3: If $\xi_0 = \xi_1$, there is no terms-of-trade externality.

c) The collateral always externality creates over-investment ex-ante ($k_0^{CE} > k_0^P$) and over-selling ex-post ($k_1^{CE} < k_1^P$).

[16] I describe in the appendix how to define the planner's problem to avoid unbounded transfers.

Proof. See appendix. □

The first lines in equations (5) and (6), which characterize the planner's solution, are identical to equations (2) and (3), which characterize the competitive equilibrium. The difference between the planner's solution and the competitive solution comes from the second line in equations (5) and (6), which contains the two distinct externalities. I refer to the first externality as the terms-of-trade externality and to the second one as the collateral externality.

The **terms-of-trade externality** arises because experts do not internalize that by increasing k_0^P at the margin, they will have to sell more units of capital at $t = 1$, which will reduce the price by $F''\left(k_0^P - k_1^P\right)$ of all the (inframarginal) units of capital sold $k_0^P - k_1^P$. This reduction in the amount received by the experts when selling needs to be multiplied by $\frac{\lambda_1}{\lambda_0} - 1$, to be expressed in $t = 0$ dollars after subtracting the compensation that needs to be given to the financiers. By construction, $k_0^P - k_1^P$ is strictly positive and $F''\left(k_0^P - k_1^P\right)$ is strictly negative, so the sign of the terms-of-trade externality depends exclusively on the sign of $\left(\frac{\lambda_1}{\lambda_0} - 1\right)$.

When $\frac{\lambda_1}{\lambda_0} > 1$, experts value resources more at $t = 1$ relative to financiers, so the externality term is negative in equation (5), increasing the marginal cost of increasing k_0 from the perspective of the planner and reducing the marginal cost of increasing k_1 (selling less). Intuitively, it is optimal for the planner to reduce k_0 and increase k_1, reducing the fire sale and hence redistributing resources towards experts in state $t = 1$ and compensating financiers with a transfer at $t = 0$.

When $\frac{\lambda_1}{\lambda_0} < 1$, experts value having resources more at $t = 0$ relative to financiers, so the externality term is positive in equation (5). Intuitively, it is optimal for the planner to increase k_0 and reduce k_1, making prices even lower in the fire sale and redistributing resources to experts in state $t = 0$.

When $\frac{\lambda_1}{\lambda_0} = 1$, the marginal rates of substitution between experts and financiers are equal. In that case, markets are effectively complete between $t = 0$ and $t = 1$, and there is no scope for any welfare improving intervention.

These results show that terms-of-trade externality can induce both over and under-investment. It may seem that the terms-of-trade externality can only generate over-investment when experts are savers in equilibrium, which may seem counterfactual, since most natural holders of assets are also natural borrowers. This is not necessarily the case. When there is uncertainty, experts may be net borrowers at the same time that the terms-of-trade externality creates over-investment — see the results in section 5. However, for over-investment to occur, it must be the case that experts would like to hedge (that is, they would like to arrange insurance for a given state) the fire sale risk if they had the possibility.

The **collateral externality** arises because experts do not internalize that selling an additional unit of capital depresses the equilibrium price and, consequently, reduces the borrowing capacity of other constrained experts. Formally, because the multiplier in the collateral constraint ν_1 is strictly positive, the term corresponding to the collateral externality is positive in the planner's Euler equation for k_0 (negative in the one for k_1). Hence, the planner perceives an additional cost of reducing prices when increasing k_0/decreasing k_1, generating over-investment ex-ante and over-selling ex-post.

There are important practical takeaways from differentiating both types of externalities. On the one hand, the Pareto improving intervention that corrects the terms-of-trade externality simply redistributes resources between different agents through price changes; the fact that their marginal rates of substitutions across periods were not equalized is what creates the scope for welfare improvements. On the other hand, the Pareto improving intervention that corrects the collateral externality simply seeks to increase prices to boost borrowing capacity. Generally, the relative strength of terms-of-trade externalities and collateral externalities determines whether there exists over- or under-investment in the constrained optimum with transfers with respect to the competitive equilibrium.

3.2 Ex-ante transfers not allowed

I now analyze the case in which the planner cannot transfer resources between experts and financiers at $t = 0$, reducing the set of instruments available to the planner. The main result of this analysis is that there are no feasible Pareto improvements, so the decentralized allocation is necessarily efficient when the planner cannot use ex-ante transfers.

Proposition 2. (Planner's solution without ex-ante transfers) *When ex-ante transfers are not allowed, there are no constrained Pareto improvements.*

Proof. See appendix. □

Intuitively, let's assume that the planner could perturb the value of k_0 at the competitive equilibrium by dk_0 (the logic is identical if he varies k_1 by dk_1; see the appendix). Using the envelope theorem, we can write the changes induced in the indirect utility of experts and financiers as:

$$dV^e = \left[\lambda_1 \left(k_0 - k_1\right) + \nu_1 \phi_1 k_1\right] \frac{dp_1}{dk_0} dk_0$$
$$dV^f = -\left(k_0 - k_1\right) \frac{dp_1}{dk_0} dk_0,$$

where $\frac{dp_1}{dk_0} = F''\left(k_0 - k_1\right)$ internalizes the effect of the change on equilibrium prices. The way to show that there are no possible improvements in this situation is by noting that the ratio of changes in indirect utilities has to be negative for any dk_0, implying that dV^e and dV^f cannot be positive for both experts and financiers at the same time. Formally:

$$\frac{dV^e}{dV^f} = -\frac{\lambda_1 (k_0 - k_1) + \nu_1 \phi_1 k_1}{(k_0 - k_1)} < 0$$

What is the intuition behind the absence of Pareto improvements? Let's say that the planner reduces k_0, inducing experts to sell less capital at $t = 1$ and thus raising the equilibrium price. This policy increases experts welfare in two different ways. First, they get a better deal from their sale, because prices are higher. Second, because prices are higher, they can borrow more per unit of capital. However, financiers always lose, because they get a worse deal from their purchase of capital, i.e., the terms-of-trade in their purchase have worsened. The opposite occurs when dk_0 is negative, which implies that the equilibrium is efficient without ex-ante transfers.

Note that, if the planner could redistribute resources at $t = 0$, financiers would still have worse terms-of-trade at $t = 1$, i.e., they would be making a worse deal, but now they could be compensated with an ex-ante transfer. This is the situation analyzed in the previous section.

3.3 Remarks

I highlight several implications of propositions 1 and 2 in a series of novel remarks. These remarks, which contain important takeaways for practical policymaking, can only be drawn when analyzing both types of externalities within a unified framework.

Remark 1. *(Flows versus stocks)*

The magnitude of the terms-of-trade externality depends on the flow $k_0 - k_1$ of units of capital sold. On the contrary, the magnitude of the collateral externality depends on the full stock of collateralizable asset held by experts k_1. Intuitively, the terms-of-trade externality requires that assets change hands in equilibrium, since it is the transfer of resources through price changes what creates the externality. The collateral externality affects any agent that uses the stock of the asset as collateral, even when he is not buying or selling it. Therefore, price changes that affect the value of the stock of assets used as collateral, e.g., houses or some types of financial securities, do matter for collateral externalities, while price changes that redistribute wealth between buyers and sellers are important for the terms-of-trade externality. Hence, the relative importance of one type of externality versus the other for a given asset depends on how much such asset is used as collateral relative to how much it is traded.

Remark 2. *(Representative agent)*

In case 3 in proposition 2b), marginal rates of substitution across periods are equalized, so markets are effectively complete for welfare purposes. In that case, the terms-of-trade externality cannot arise, but the collateral externality is still present. This configuration is implicit in models in which welfare is calculated using a representative agent approach, for instance, Bianchi (2011), Jeanne and Korinek (2010) or Stein (2012). These papers find Pareto improvements even without ex-ante transfers, because they only have collateral externalities and calculate welfare using a representative agent.[17]

Remark 3. *(Zero weight on financiers)*

The analysis so far has been very careful in defining Pareto improvements. In particular, the lack of Pareto improvements when there are no ex-ante transfers is due to the fact that the planner is unable to compensate financiers for having higher prices in the fire sale. In an interesting alternative environment, which may be a better representation of actual policy discussions, the planner could exclusively maximize the welfare of experts, giving zero weight to the utility of financiers. In that case, the externality term in equations (5) and (6) can be written as:

$$\left(\frac{\lambda_1}{\lambda_0}\right)\left(k_0^{PT}-k_1^{PT}\right)F''\left(k_0^{PT}-k_1^{PT}\right)+\frac{\nu_1}{\lambda_0}\phi_1 k_1^{PT}F''\left(k_0^{PT}-k_1^{PT}\right)<0,$$

which is a strictly negative expression. Hence, a planner who cares more about experts than financiers perceives that there is always over-investment at $t=0$ and over-selling at $t=1$. Intuitively, this is the case of a monopolist expert that fully internalizes that selling more capital at $t=1$ reduces the equilibrium price and this lowers his welfare in two different ways: by getting a worse deal when selling capital and by reducing his borrowing capacity.

Remark 4. *(Why generic inefficiency results need not apply)*

The reader may wonder why the generic constrained inefficiency results shown by Geanakoplos and Polemarchakis (1986) and Greenwald and Stiglitz (1986) do not apply here unless the planner can use ex-ante transfers. The crucial issue is that the fire sale occurs in a single period/state, while those results require rank conditions to hold, i.e., sufficiently many assets, states, periods and/or goods are required to find generic improvements. With more periods/states, it would be possible to modify allocations in such a way that changes in prices in different periods/states compensate each other in the right way to induce Pareto improvements, even without using ex-ante transfers.

The practical takeaway from this result is that, if fire sales are events that happen only in a particular state — that is the common assumption in macro-finance models[18] — ex-post Pareto

[17]In international models, as Bianchi (2011), it does not matter whether the planner accounts for the welfare of perfectly competitive foreign financiers, because they make zero profit in equilibrium and remain indifferent across allocations.

[18]Take, as a recent example, the study of binomial economies by Fostel and Geanakoplos (2014), who argue that it is sensible to study leverage and price changes in a two-state environment.

improvements require that the planner uses additional instruments. Consequently, the generic inefficiency results lose their bite when fire sales occur in a single state/period.

Remark 5. *(Decentralization)*

In this simple environment, decentralizing the constrained efficient allocation with taxes on the purchases or sales of capital is straightforward. In more general environments, individual specific taxes will be needed.

In the cases in which the planner wants to increases prices at $t = 1$, he can implement the constrained efficient allocation by taxing initial purchases of capital, which in this model can be interpreted as a form of "capital requirements", and subsidizing purchases of capital by financiers at $t = 1$, which can be interpreted as a policy of "asset purchases/lender of last resort".

Remark 6. *(Binding constraints versus lack of assets as sources of market incompleteness)*

The environment analyzed in the paper embeds, as a special case, the situation in which experts have no access to financial markets between periods. For instance, when $\phi_t = S_t = 0$, experts cannot borrow or save, so market are fully incomplete — this is exactly the formulation studied in Geanakoplos and Polemarchakis (1986). Obviously, in those situations there cannot be collateral externalities because there are no intertemporal markets but, importantly, the effect of the terms-of-trade externality in equations (5) and (6) remains identical. This argument shows that terms-of-trade externalities arise when markets are effectively incomplete, independently of the source of incompleteness, which could either be a missing market or a binding constraint in an existing market.

4 Additional results

This section studies a) the relation between the efficiency results derived above and the presence of amplification mechanisms and b) the time consistency of the constrained efficient policy.

4.1 Amplification and welfare

A vast literature has shown that financial constraints can be a powerful amplification mechanism of shocks — see Brunnermeier, Eisenbach and Sannikov (2011) for a recent survey. Although amplification mechanisms can be relevant quantitatively, I show that feedback loops between prices and the amount of assets sold are neither necessary nor sufficient for an equilibrium to be inefficient, which implies that normative and positive implications of fire sales must be decoupled.

I exclusively modify the baseline model by assuming that the endowment of experts at $t = 1$ is sufficiently negative ($w_1 \ll 0$) so that $c_1 = 0$. This assumption can be understood as experts experiencing a (perfectly foreseen) negative income shock at $t = 1$. Alternative assumptions implying $c_1 = 0$, e.g., a sudden tightening of borrowing constraints ($\phi_1 \ll \phi_0$), would deliver the same results.

When $c_1 = 0$, this assumption implies that the experts' budget constraint at $t = 1$ can be written as:

$$\underbrace{p_1 (k_1 - k_0)}_{\text{Amount sold}} = \frac{1}{1 - \phi_1} \left[\underbrace{w_1 + G_1 (k_0)}_{\text{Initial net worth}} + \underbrace{(\phi_1 p_1 - \phi_0 p_0) k_0}_{\text{Change borrowing capacity}} \right] \tag{7}$$

Equation (7), when combined with the pricing equation $p_1 = F' (k_1 - k_0)$, yields a pair of downward sloping equations in the space $(p_1, k_0 - k_1)$ — see the right plot in figure 2.[19] Intuitively, imagine that there is an negative shock to w_1. To prevent consumption from being negative, experts must sell capital and deleverage, lowering prices, which increases the need to sell capital and further reduces prices. Lower prices increase the need to sell more capital and so on. I use the term amplification to refer to this mechanism in which the slopes of the supply and demand for capital have the same sign, magnifying small shocks to net worth. Note that now $\lambda_1 = \xi_1 + \chi_1$, with $\chi_1 > 0$.

Proposition 3 presents the main theoretical result of this section.

Proposition 3. (Welfare in model with amplification) *When the planner has access to ex-ante transfers, the constrained planner's Euler equation for k_0 is determined by equations (5) and (6), as in proposition 1. All normative implications from the baseline model extend to this environment.*

Proof. See appendix. □

Proposition 3 shows that the same equations that characterize the constrained optimum in the baseline model, in which there is no amplification, are identical to those that characterize the constrained policy when shocks to net worth get amplified. This comparison shows that the presence of feedback between prices and selling motives is neither necessary nor sufficient for the existence of externalities.

The best way to understand this decoupling result is graphically using figure 2.[20] Both plots in the figure show the relation between the price of capital at $t = 1$ and the amount of capital sold. The left plot shows the diagram in the baseline model, in which there is no

[19] Depending on the primitives, multiple equilibria could exist in this environment. For simplicity, I implicitly assume that we are in a unique equilibrium situation.

[20] Using both Euler equations in (1), it is easy to show that, in the baseline model, $\frac{dk_0}{dp_1} > 0$ and $\frac{dk_1}{dp_1} < 0$, which guarantees that the red dashed line in the left plot of figure 2 is upward sloping. Intuitively, experts invest more capital at $t = 0$ and sell more capital at $t = 1$ when the price of capital is high.

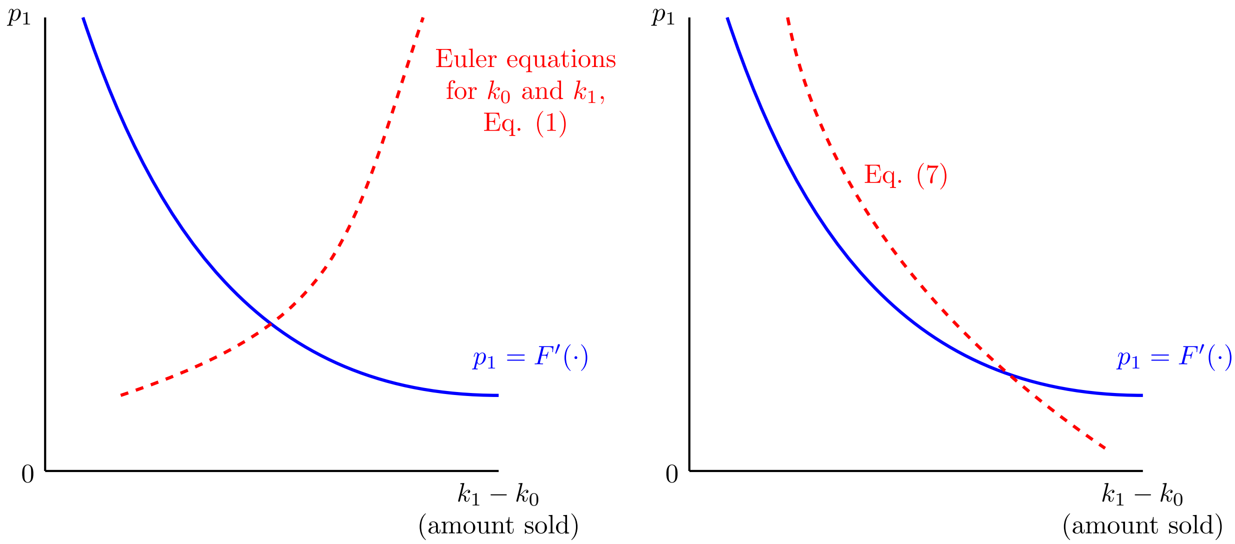

Figure 2: Left plot: no amplification (baseline model). Right plot: amplification (this section).

amplification, and the right plot shows the same diagram in the model of this section, which features amplification. The result in proposition 3 shows that the normative analysis from the baseline model extends without modification to the model with amplification. Therefore, an economy with amplification can be efficient or inefficient, using the same arguments that apply to an economy without amplification.

Decoupling amplification and welfare matters because observing fire sales in which low prices and asset sales reinforce each other does not justify a government intervention per se. Government intervention must address the particular market failure (wedges) created by the terms-of-trade and collateral externalities identified in this paper. Unfortunately, arguing that amplification mechanisms cause fire sale externalities is a common misconception in many discussions of these issues. This result does not imply that amplification and pecuniary externalities are unrelated phenomena — they may appear jointly in an environment like the one studied in this paper and the quantitative relevance of the externalities will be larger if there are amplification mechanisms that generate deeper fire sales — but it says that the normative discussions should focus on the wedges caused by the externality terms.

4.2 Time inconsistency

I now show that the optimal constrained efficient policy (with ex-ante transfers) characterized in section 3 is in general time inconsistent. I also characterize the time consistent policy.[21]

[21] There existed no work on time inconsistency before the first version of this paper was disseminated. See Jeanne and Korinek (2012) and Bianchi and Mendoza (2013) for recent developments along those lines.

When externalities are present, the constrained efficient allocation from a $t = 0$ perspective entails correcting both k_0 and k_1. I now assume that the planner has the possibility of re-optimizing at $t = 1$. To be coherent with the notion of constrained efficiency, the planner cannot use transfers at $t = 1$.[22] Under those assumptions, proposition 4 follows.

Proposition 4. (Time inconsistency)

a) The constrained efficient policy is time inconsistent.

b) The time consistent choice of k_0 is characterized by:

$$k_0 : p_0 = \frac{\lambda_1}{\lambda_0}\left(G_1'\left(k_0^{PC}\right) + F'\left(k_0^{PC} - k_1^{PC}\right)\right) + \frac{\nu_0}{\lambda_0}\phi_0 p_0 + \left(1 - \frac{d\tilde{k_1}\left(k_0^{PC}\right)}{k_0}\right)\Omega, \tag{8}$$

where $\tilde{k_1}\left(k_0\right)$ is the equilibrium mapping between $t = 1$ choices of capital given the holdings of k_0. It is shown in the appendix that $0 < \frac{d\tilde{k_1}(k_0)}{k_0} < 1$. The expression for Ω is defined as:

$$\Omega \equiv \left(\frac{\lambda_1}{\lambda_0} - 1\right)\left(k_0^{PC} - k_1^{PC}\right) F''\left(k_0^{PC} - k_1^{PC}\right) + \frac{\nu_1}{\lambda_0}\phi_1 k_1^{PC} F''\left(k_0^{PC} - k_1^{PC}\right)$$

Proof. See appendix. □

Intuitively, at the initial period, the planner designs a set of policies that a) help to complete the market and b) alleviate the collateral frictions. However, for a planner without transfers, the economy is efficient from a $t = 1$ perspective, using the results derived in proposition 2. Hence, a planner who re-optimizes at $t = 1$ has no desire to introduce any distortion on markets, which results in a time inconsistency policy.[23]

The time consistent planning solution takes into account the equilibrium mapping $\tilde{k_1}\left(k_0\right)$, which is determined by the Euler equation for k_1 in the competitive market — given in equation (3). There are two interesting benchmarks. On the one hand, if $\frac{d\tilde{k_1}(\cdot)}{dk_0} \approx 1$, the time consistent solution replicates the one of the competitive economy, since any effect on prices induced by varying k_0 is undone by investors by modifying k_1. Alternatively, when $\frac{d\tilde{k_1}(\cdot)}{k_0} = 0$, the time consistent Euler equation is identical to the discretionary one at $t = 0$, which must be combined with equation (3) to fully determine the solution. Note that equation (8) would also characterize the optimal policy of a planner who is forced to use exclusively $t = 0$ policies.

Unfortunately, it is not possible to determine in general whether the time consistent choice of k_0 is higher or lower than the one chosen in the constrained efficient solution. The intuition for

[22]Once again, the choice of welfare benchmark is not straightforward. I briefly discuss below how the results would change when the planner has access to a transfer at $t = 1$.

[23]If the equilibrium price q_t had a forward looking component, there would be an additional channel — not analyzed here — that may create time inconsistency. In that case, the the planner would commit at period t to future policies that raise the price q_t, increasing period t borrowing capacity. However, at later stages, there is no reasons for the planner to respect those commitments, for similar reasons to those described in this section.

this result is that k_1 now depends endogenously on k_0 and the equilibrium price is determined by its difference $k_0 - k_1$. If k_1 reacts more than proportionally with k_0, it may be optimal to have a larger k_0 with respect to the constrained efficient, since $k_0 - k_1$ will be smaller and the fire sale reduced.

In practice, policies that seek to correct fire sales externalities will suffer from time inconsistency problems. This is unavoidable because of the intertemporal and across-state nature of these externalities.

5 Extensions

This section extends the insights from the baseline model to more general environments with risk averse agents, uncertainty and collateral constraints with price dependent margins.

5.1 Risk aversion and uncertainty

The baseline model assumes that there is no uncertainty and that experts' utility is linear on units of the consumption good at each period. I now assume that there is uncertainty at $t = 1$, that experts are risk averse and that there are two groups of financiers.[24] One group of financiers is risk neutral and issues/purchases state contingent securities to/from experts demanding a risk-free return R, subject to the credit constraints described below. A second group of financiers is risk averse and purchases the capital sold at $t = 1$. In particular, I assume that the utility of experts and the utility of the risk averse financiers are respectively given by U^e and U^f:[25]

$$U^e = U(c_0) + \mathbb{E}\left[U_s(c_{1s}) + U_s(c_{2s})\right] \quad \text{and} \quad U^f = \tilde{U}(c_0) + \mathbb{E}\left[\tilde{U}_s(c_{1s}) + \tilde{U}_s(c_{2s})\right]$$

There is only uncertainty at $t = 1$, with $s = \{1, 2, \ldots, S\}$ possible states of nature that occur with probability π_s. I assume that all primitives of the model are stochastic: endowments, production technologies, preferences and borrowing limits. Experts can issue/purchase state contingent securities from the risk neutral financiers subject to state-by-state credit constraints:[26]

$$-\phi_{0s} p_0 k_0 \leq a_{0s} \quad \text{and} \quad a_{0s} \leq S_{0s},\ s = \{1, \ldots, S\}$$

$$-\phi_{1s} p_{1s} k_{1s} \leq a_{1s} \quad \text{and} \quad a_{1s} \leq S_{1s},\ s = \{1, \ldots, S\}$$

[24] The presence of risk neutral financiers is required to pin down the price of the state contingent securities. If those securities were priced by risk averse agents, the planner would try to affect the equilibrium price of those assets in general equilibrium, since there would exist a new set of terms-of-trade externalities.

[25] The utility of the risk neutral financiers is irrelevant for welfare purposes.

[26] But for the timing difference regarding p_t and p_{t+1}, Rampini and Viswanathan (2010) show that this type of credit constraints can be microfounded by combining complete markets and limited commitment.

I also assume that there are only downward sloping demands for capital when sold, that is, experts can purchase new capital at $t=1$ at a constant price $\overline{p}$. To make the problem smooth, I relax the Inada conditions on $F_s(\cdot)$ and assume that $\lim_{k_{1s}^f \to 0} F_s'\left(k_{1s}^f\right) = \overline{p}$. Therefore, $F_s''(\cdot) = 0$ in all states in which experts do not sell capital.

I focus on the solution of the planner's problem with ex-ante transfers; the extensions are straightforward.

Proposition 5. (Planner's solution with risk averse agents and uncertainty) *When the planner has access to ex-ante transfers, the planner's Euler equation for k_0 is determined by:*

$$p_0 = \mathbb{E}\left[\frac{\lambda_{1s}^e}{\lambda_0^e}\left(G_{1s}'\left(k_0^P\right) + F_s'(\cdot)\right) + \frac{\nu_{0s}^e}{\lambda_0^e}\phi_{0s}p_0\right] \tag{9}$$

$$+\mathbb{E}\left[\underbrace{\left(\frac{\lambda_{1s}^e}{\lambda_0^e} - \frac{\lambda_{1s}^f}{\lambda_0^f}\right)\left(k_0^P - k_{1s}^P\right)F_s''(\cdot)\,\alpha_s}_{\text{Terms-of-trade}} + \underbrace{\frac{\nu_{1s}^e}{\lambda_0^e}\phi_{1s}k_{1s}^P F_s''(\cdot)\,\alpha_s}_{\text{Collateral}}\right],$$

where λ_{ts}^e and λ_{ts}^f equal the marginal utility of income for experts and financiers and $\alpha_s = \mathbb{I}_s\left[k_0^P - k_{1s}^P > 0\right]$ is an indicator function that is active when experts sell capital. See the appendix for the Euler equations for k_{1s}^P, which contain identical externality terms.

Proof. See appendix. □

Equation (9) shows that the insights drawn in the risk neutral case do not go away when agents are risk averse. When experts sell assets in fire sale states, $(k_0 - k_{1s}) F_s''(\cdot) < 0$, the planner perceives a higher marginal cost of increasing k_0 when experts marginal utility of income is relatively larger with respect to financiers, that is, when $\frac{\lambda_{1s}^e}{\lambda_0^e} > \frac{\lambda_{1s}^f}{\lambda_0^f}$, In those cases, the terms-of-trade externality term is negative. The terms-of-trade externality term is positive when the opposite configuration of relative MRS occurs, that is, when $\frac{\lambda_{1s}^e}{\lambda_0^e} < \frac{\lambda_{1s}^f}{\lambda_0^f}$ in fire sale states.[27] Equation (9) makes clear that the terms-of-trade externality occurs as long as marginal rates of substitution are not equalized. When markets are complete, $\frac{\lambda_{1s}^e}{\lambda_0^e} = \frac{\lambda_{1s}^f}{\lambda_0^f}$, and the terms-of-trade externality goes away.

When borrowing constraints bind at $t=1$, ν_{1s}^e is positive and the planner also perceives a higher cost of increasing k_0. As in the risk neutral case, the collateral externality term is always

[27] At this level of generality, it is not possible to characterize explicitly how k_0^P and $\left\{k_{1s}^P\right\}_s$ differ from the competitive outcome. It is natural to expect that the planner's solution will feature over-investment in k_0 when the externality term in equation (9) is negative and under-investment when it is positive, but strong income effects could occasionally overturn these results.

negative, pushing towards over-investment. It is natural to expect ν_{1s}^{e} to be positive, which implies that experts borrow as much as they can to avoid the fire sale of capital.

Finally, note that equation (9) would still hold if we had assumed that agents traded cannot trade contingently across all states — for instance, by assuming that $a_{0s} = a_{0s'}$ for a pair of states s and s', by restricting experts to trade a single noncontingent bond or by eliminating all financial markets altogether. Additional restrictions on the set of asset trades only modify the determination of λ_{ts}^{e} in equilibrium, but equation (9) remains valid. In general, the set of tradable assets is crucial to determine the effects of terms-of-trade externalities.[28]

The stochastic formulation avoids the counterintuitive implication of the baseline model implying that the terms-of-trade externality can only generate over-investment when experts are savers in equilibrium. The robust prediction of the model is that, for over-investment to occur, experts would like to buy insurance that pays off in fire sale states at ongoing market prices.

5.2 Price dependent margins/general binding-price-constraint externalities

The baseline model assumes that the dollar amount borrowed by experts' is bounded by a constant fraction of the amount of capital held. Let's now assume instead that the maximum amount that experts can borrow is given by a fraction $\Phi_t(p_t) \in [0, 1)$, which may depend on equilibrium prices, of the total value of the collateral $p_t k_t$. That is:

$$-\Phi_t(p_t)\, p_t k_t \leq a_t$$

In the baseline model, $\Phi_t(p_t) = \phi_t$, a constant. This type of constraint can emerge endogenously in models in which lenders determine borrowing capacity depending on equilibrium prices. More generally, this specification captures in reduced form other contractual frictions in which prices determine borrowing capacities endogenously, as in models with imperfect information or asymmetric information. Brunnermeier and Pedersen (2009) and, especially, Gersbach and Rochet (2013), who execute a normative analysis, are good examples.[29]

I focus on the solution of the planner's problem with ex-ante transfers; the extensions are straightforward.

[28] Restricting the set of assets traded can deliver any relation between the status of experts as lenders/borrowers and the sign of the terms-of-trade externality term. For instance, experts who trade a non-contingent bond may be net borrowers but also value greatly having resources in fire-sale states.

[29] In Gersbach and Rochet (2013), banks allocate resources at a initial stage between good and bad aggregate states. They fail to internalize that their ex-ante hedging decisions change the price of capital at an interim stage which, through its effects on a binding credit constraint, distorts the allocation of resources between the traditional sector and the (more efficient) banking sector.

Proposition 6. (Planner's solution with price dependent margins) *When the planner has access to ex-ante transfers, the externality term, equivalent to the one defined in equations (5) and (6), becomes:*

$$\underbrace{\left(\frac{\lambda_1}{\lambda_0}-1\right)\left(k_0^P-k_1^P\right)F''\left(k_0^P-k_1^P\right)}_{\textit{Terms-of-trade}}$$
$$+\underbrace{\frac{\nu_1}{\lambda_0}\Phi_1\left(\cdot\right)k_1^P F''\left(k_0^P-k_1^P\right)}_{\textit{Collateral}}+\underbrace{\frac{\nu_1}{\lambda_0}\Phi_1'\left(\cdot\right)F'\left(k_0^P-k_1^P\right)k_1^P F''\left(k_0^P-k_1^P\right)}_{\textit{Margin}}$$

Proof. See appendix. □

The first two terms are identical as those in equation (5). The last term is the new externality. Intuitively, experts do not internalize that when they sell capital, they tighten the margins of other agents who are borrowing constraints. When $\Phi'(\cdot)>0$, high prices are associated with high borrowing capacity. In that case, the margin/binding-price-constraint externality has the same sign as the collateral externality, implying that experts over-invest at $t=0$ and over-sell at $t=1$, which seems the empirically relevant case — see, for instance, Adrian and Shin (2010). When $\Phi'(\cdot)<0$, high prices are associated with low borrowing capacity. In that case, the margin/binding-price-constraint externality pushes in the opposite direction to the collateral externality. At least theoretically, a margin/binding-price-constraint externality can cause over- or under-investment.

The **margin or binding-price-constraint** externality is of the same nature as the collateral externality, because price changes directly affect the borrowing capacity of agents in the economy. As the collateral externality, it can be traced back to Greenwald and Stiglitz (1986), but it is unrelated to the mechanism present in Geanakoplos and Polemarchakis (1986). A mixture of the three externalities, terms-of-trade, collateral and margin/binding-price-constraint, determines the overall optimal policy.

6 Conclusion

This paper has shown that two distinct externalities, terms-of-trade externalities and collateral externalities, arise in Walrasian models with incomplete markets and/or credit constraints. A single friction — a credit (collateral) constraint — can create two different externalities. Policies aimed at improving intertemporal hedging or risk-sharing through terms-of-trade changes or those aimed at boosting borrowing capacity can improve welfare in this environment, although ex-ante transfers are generally needed to reach Pareto improvements.

It should not be sufficient for future work to argue that a paper contains a fire sale/pecuniary externality, it will be important to identify which type of externality — either of the terms-of-trade type or the collateral/binding-price-constraint type — is present. Future work should also acknowledge that the analysis of amplification mechanisms and pecuniary externalities, even if both caused by a same set of credit constraints, must be done separately. The existence of amplification mechanisms is neither necessary nor sufficient for fire sales/pecuniary externalities to exist.

The results of this paper will discipline academic and policy discussions on the role of pecuniary externalities as a rationale for financial regulation and macro-prudential policy.

Appendix: Proofs and derivations

Proofs: Section 2

Experts' problem

The Lagrangian corresponding to the problem solved by experts is given by:

$$\begin{aligned}\mathcal{L}^e = \xi_0 c_0 + \xi_1 c_1 + \xi_2 c_2 - \lambda_0 \left(c_0 + p_0 k_0 + a_0 - w_0\right) \\ -\lambda_1 \left(c_1 + p_1 k_1 + a_1 - w_1 - G_1\left(k_0\right) - p_1 k_0 - R a_0\right) - \lambda_2 \left(c_2 - w_2 - G_2\left(k_1\right) - R a_1\right) \\ +\nu_0 \left(a_0 + \phi_0 p_0 k_0\right) - \eta_0 a_0 + \nu_1 \left(a_1 + \phi_1 p_1 k_1\right) - \eta_1 a_1 + \chi_0 c_0 + \chi_1 c_1 + \chi_2 c_2\end{aligned} \tag{10}$$

With optimality conditions given by:

$$\begin{aligned} c_t : \lambda_t = \xi_t + \chi_t,\ t = 0,1,2 \qquad &\text{and} \qquad a_t : \lambda_t = R\lambda_{t+1} + \nu_t - \eta_t,\ t = 0,1 \\ k_0 : p_0 \lambda_0 = \lambda_1 \left(G_1'\left(k_0\right) + p_1\right) + \nu_0 \phi_0 p_0 \qquad &\text{and} \qquad k_1 : p_1 \lambda_1 = \lambda_2 G_2'\left(k_1\right) + \nu_1 \phi_1 p_1\end{aligned}$$

I do not impose the non-negativity constraints directly, as in the financiers problem, for easier comparison with the results of section 4.

Financiers' problem

Substituting directly the budget constraints, given the assumption of large endowments, the Lagrangian corresponding to the financiers' problem becomes:

$$\mathcal{L}^f = \overbrace{w_0^f + a_0^f}^{c_0^f} + \overbrace{w_1^f + a_1^f - R a_0^f + F\left(k_1^f\right) - p_1 k_1^f}^{c_1^f} + \overbrace{w_2^f - R a_1^f}^{c_2^f}$$

With optimality conditions given by:

$$R = 1 \ \text{ and } \ p_1 = F'\left(k_1^f\right) = F'\left(k_0 - k_1\right)$$

Lemma 1. (Competitive equilibrium)

Substituting the values of λ_t into the Euler equations (noting that $\chi_t = 0$) for capital, rearranging and substituting $p_1 = F'\left(k_0^{CE} - k_1^{CE}\right)$, we can write:

$$p_0 \left(\xi_0 - \nu_0 \phi_0\right) = \xi_1 \left(G_1'\left(k_0^{CE}\right) + F'\left(k_0^{CE} - k_1^{CE}\right)\right)$$

$$F'\left(k_0^{CE} - k_1^{CE}\right)\left(\xi_1 - \nu_1 \phi_1\right) = \xi_2 G_2'\left(k_1^{CE}\right)$$

Substituting $\xi_1 - \nu_1 \phi_1 = \xi_1 \left(1 - \phi_1\right) + \xi_2 \phi_1$ yields equations (2) and (3) in the paper.

We can rearrange both equations as:

$$p_0 \frac{\xi_0 - \nu_0 \phi_0}{\xi_1} = G_1'\left(k_0^{CE}\right) + \frac{\xi_2}{\xi_1 \left(1 - \phi_1\right) + \xi_2 \phi_1} G_2'\left(k_1^{CE}\right) \tag{11}$$

$$F'\left(k_0^{CE} - k_1^{CE}\right) = \frac{\xi_2}{\xi_1 \left(1 - \phi_1\right) + \xi_2 \phi_1} G_2'\left(k_1^{CE}\right) \tag{12}$$

Note that equation (11) yields an downward sloping relation in the (k_0, k_1) space while equation (12) implies an upward sloping relation; since both equations are continuous, uniqueness is guaranteed. Graphically, in figure A.1, with the 45 degree line as a reference:

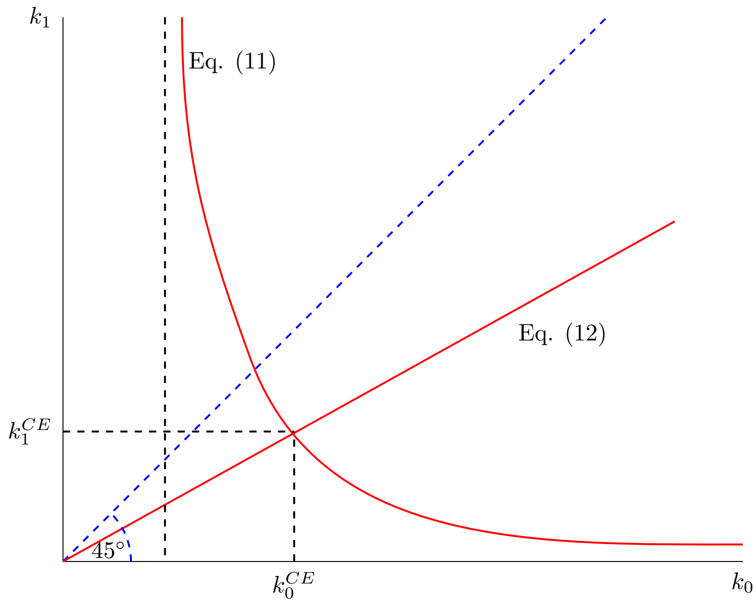

Figure A.1: Competitive equilibrium

Once k_0^{CE} and k_1^{CE} are determined, the equilibrium price is given by $p_1 = F'\left(k_0^{CE} - k_1^{CE}\right)$. Depending on the values of ξ_t, at most one of the credit constraints will bind per period, what pins down a_t — in the case that no constraint binds at period t, there is a range of a_t that are consistent with a given equilibrium. Once a_t is determined, consumption can be found directly from the budget constraints.

Example 1. (Logarithmic production)

Substituting the functional form implied by logarithmic production in equation (2):

$$\frac{1}{k_0^{CE} - k_1^{CE}}\left(\xi_1 - \left(\xi_1 - \xi_2\right)\phi_1\right) = \frac{\xi_2}{k_1^{CE}} \Rightarrow k_0^{CE} = \left[1 + \frac{\xi_1\left(1-\phi_1\right) + \xi_2\phi_1}{\xi_2}\right] k_1^{CE}$$

By rearranging this equation we can find equation (4) in the paper. Substituting also the functional form implied by logarithmic production in equation (3), we can find a close form solution for k_1^{CE}.

$$k_1^{CE} = \frac{\xi_1}{p_0\left(\xi_0 - \nu_0\phi_0\right)}\left(\frac{1}{1 + \frac{\xi_1(1-\phi_1)+\xi_2\phi_1}{\xi_2}} + \frac{1}{\frac{\xi_1(1-\phi_1)+\xi_2\phi_1}{\xi_2}}\right)$$

Proofs: Section 3

Proposition 1. (Planner's solution with ex-ante transfers)

a) The Pareto frontier for a constrained planner is characterized by:

$$\max U^e + \theta U^f$$

subject to, budget, credit and non-negativity constraints, once equilibrium prices and market clearing conditions are imposed.[30] The choice of θ parametrizes the Pareto frontier between experts and financiers. Note however, that given the linearity of the utility specifications and the possibility of using lump-sum ex-transfers, by choosing $\theta = \xi_0$, we are characterizing the planning problem that maximizes total surplus. Alternatively, we could have divided U^e by ξ_0 (this would be simply a normalization), and then guarantee that the constrained planner is maximizing total surplus using ex-ante transfers by solving $\max \frac{U^e}{\xi_0} + U^f$. Following this approach avoids the problem of having to bound the amount of ex-ante transfers or making them costly.

Using this approach, the Lagrangian corresponding to the planner's problem is:

$$\begin{aligned}\mathcal{L} = \xi_0 c_0 + \xi_1 c_1 + \xi_2 c_2 - \lambda_0 \left(c_0 + p_0 k_0 + a_0 - w_0 + T_0\right) - \eta_0 a_0 + \nu_0 \left(a_0 + \phi_0 p_0 k_0\right) \\ -\lambda_1 \left(c_1 + F'\left(k_0 - k_1\right)\left(k_1 - k_0\right) + a_1 - w_1 - G_1\left(k_0\right) - R a_0\right) \\ -\eta_1 a_1 + \nu_1 \left(a_1 + \phi_1 F'\left(k_0 - k_1\right) k_1\right) \\ -\lambda_2 \left(c_2 - w_2 - G_2\left(k_1\right) - R a_1\right) + \chi_0 c_0 + \chi_1 c_1 + \chi_2 c_2 \\ +\theta \left[T_0 + w_0^f + w_1^f + w_2^f + F\left(k_0 - k_1\right) - F'\left(k_0 - k_1\right)\left(k_0 - k_1\right)\right]\end{aligned} \tag{13}$$

The optimality conditions for c_t and a_t are identical to those in the problem solved by experts:

$$c_t : \lambda_t = \xi_t + \chi_t,\ t = 0,1,2 \text{ and } a_t : \lambda_0 = \lambda_1 + \nu_0 - \eta_0,\ t = 0,1$$

The optimality conditions for k_0 and k_1 are now:

$$\begin{aligned}k_0 :& \lambda_0 p_0 = \lambda_1 \left(G_1'\left(k_0\right) + F'\left(k_0 - k_1\right)\right) + \nu_0 \phi_0 p_0 \\ &\quad + \left(\lambda_1 - \theta\right)\left(k_0 - k_1\right) F''\left(k_0 - k_1\right) + \nu_1 \phi_1 k_1 F''\left(k_0 - k_1\right) \\ k_1 :& F'\left(k_0 - k_1\right)\lambda_1 = \lambda_2 G_2'\left(k_1\right) + \nu_1 \phi_1 F'\left(k_0 - k_1\right) \\ &\quad - \left[\left(\lambda_1 - \theta\right)\left(k_0 - k_1\right) F''\left(k_0 - k_1\right) + \nu_1 \phi_1 k_1 F''\left(k_0 - k_1\right)\right],\end{aligned}$$

Finally, the optimality condition for T_0 is $\theta = \lambda_0 = \xi_0$, as assumed. Substituting this condition into the previously derived optimality conditions yields equations (5) and (6) in the text.

b) The optimality conditions for k_0 and k_1 can be written as:

$$p_0 \frac{\xi_0 - \nu_0 \phi_0}{\xi_1} = G_1'\left(k_0^*\right) + \frac{\xi_2}{\xi_1\left(1 - \phi_1\right) + \xi_2 \phi_1} G_2'\left(k_1^*\right) \tag{14}$$

$$F'\left(k_0 - k_1\right) = \frac{\lambda_2}{\lambda_1 - \nu_1 \phi_1} G_2'\left(k_1\right) - \Omega, \tag{15}$$

[30] Because we are focusing on a constrained efficient benchmark, the planner must respect budget constraints, instead of directly maximizing utility subject to resource constraints.

where $\Omega \equiv (\lambda_1 - \lambda_0)(k_0 - k_1) F''(k_0 - k_1) + \nu_1 \phi_1 k_1 F''(k_0 - k_1)$. Equation (14) is identical to equation (11), which is one of the two equations that characterizes the competitive equilibrium. Note that it yields a downward sloping relation between k_0 and k_1. Equation (15) is identical to equation (12), only modified by the term Ω.[31] Figure A.2 shows the result graphically.

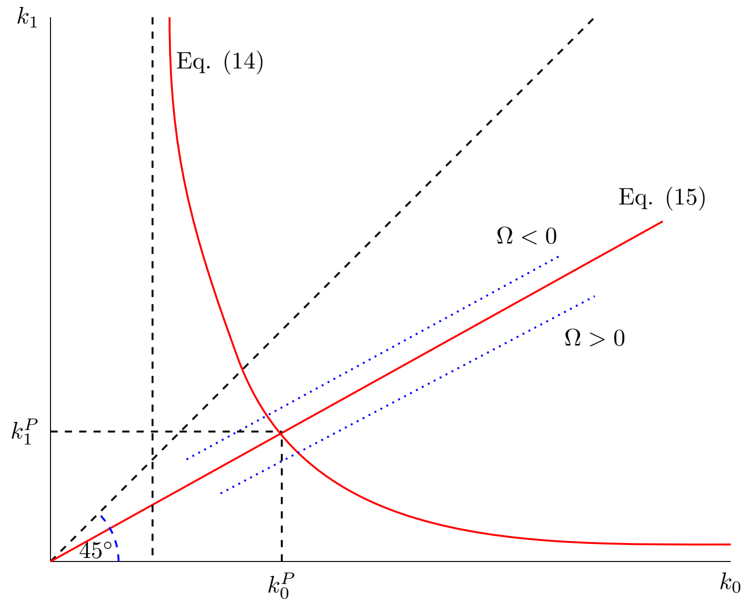

Figure A.2: Constrained optimum with ex-ante transfers

When $\Omega > 0$, we can see that $k_0^P > k_0^{CE}$. This corresponds to case 1 in proposition 2b) and proposition 2c). When $\Omega < 0$, we can see that $k_0^P < k_0^{CE}$, which corresponds to case 2 in proposition 2b). When $\Omega = 0$, which can only occur when $\lambda_1 = \lambda_0$ and $\nu_1 = 0$, the constrained efficient allocation coincides with the outcome of the competitive equilibrium.

c) See part b).

Proposition 2. (Planner's solution without ex-ante transfers)

The proof, sketched in the text, is by contradiction. Assume that there are Pareto improvement without transfers. Formally, assume that, by varying the equilibrium allocation of k_0 and $k_1$through changes dk_0 and dk_1, we can find positive increases in utility for both experts and financiers, that is:

$$dV^e > 0 \quad \text{and} \quad dV^f > 0$$

Using the envelope theorem, we can write dV^e as:

$$\begin{aligned} dV^e &= (\lambda_1 (k_0 - k_1) + \nu_1 \phi_1 k_1) \frac{dp_1}{dk_0} dk_0 + (\lambda_1 (k_0 - k_1) + \nu_1 \phi_1 k_1) \frac{dp_1}{dk_1} dk_1, \\ &= (\lambda_1 (k_0 - k_1) + \nu_1 \phi_1 k_1) \left(\frac{dp_1}{dk_0} dk_0 + \frac{dp_1}{dk_1} dk_1 \right), \end{aligned}$$

[31] Note that Ω depends on the values of k_0 and k_1, which may change the curvature of equation (15), but it does not affect the sign of the externalities. Figure A.2 depicts parallel shifts only for illustration.

and, analogously, dV^f as:

$$\begin{aligned} dV^f &= -\left(k_0 - k_1\right) \frac{dp_1}{dk_0} dk_0 - \left(k_0 - k_1\right) \frac{dp_1}{dk_1} dk_1 \\ &= -\left(k_0 - k_1\right) \left(\frac{dp_1}{dk_0} dk_0 + \frac{dp_1}{dk_1} dk_1 \right) \end{aligned}$$

Note that we can write:

$$\frac{dV^e}{dV^f} = -\frac{\lambda_1 \left(k_0 - k_1\right) + \nu_1 \phi_1 k_1}{\left(k_0 - k_1\right)} < 0,$$

but this contradicts the initial assumption that both dV^e and dV^f are strictly positive, which shows that no Pareto improvements can exist without ex-ante transfers.

Proofs: Section 4

Proposition 3. (Welfare in model with amplification) The Lagrangian described in equation (10) remains valid to describe the experts problem and, hence, the competitive equilibrium under the new set of assumptions. Note that, when $c_1 = 0$, the budget constraint reads:

$$p_1 \left(k_1 - k_0\right) = w_1 + G_1 \left(k_0\right) - \phi_0 p_0 k_0 + \phi_1 p_1 k_1, \tag{16}$$

which can be easily transformed into equation (7) in the text. Combining equation (16) with the pricing function $p_1 = F' \left(k_0 - k_1\right)$, we can see in left plot of figure 2 that it may exist the possibility of multiple equilibria in this case.

Because $c_1 = 0$, now $\lambda_1 = \xi_1 + \chi_1$. This equation for λ_1, combined with the budget constraint, the pricing function $p_1 = F' \left(k_0 - k_1\right)$ and the pair of Euler equations for k_0 and k_1 fully characterize the new competitive equilibrium. Taking as given the rest of equilibrium variables. Combining both Euler equations, and getting rid of λ_1, we can find that the following relation between the amount sold $k_0 - k_1$ and p_1:

$$p_0 \xi_0 \left(1 - \alpha_0 \phi_0\right) = \frac{\xi_2}{1 - \phi_1} \left(\frac{G_2' \left(k_1\right)}{p_1} - \phi_1 \right) \left(G_1' \left(k_0\right) + p_1 - p_0 \alpha_0 \phi_0 \right)$$

where $\alpha_0 \equiv \mathbb{I}\left[\nu_0 > 0\right]$. This equation, combined with (16), defines a system of equations for k_1 and k_0. The rest of endogenous variables are easily determined after the choices of capital are known.

The planning problem can be stated identically as in equation (13), so the set of optimality conditions is identical, which proves the statement of proposition 3. The only main difference is that now λ_1 does not equals ξ_1 in equilibrium.

Proposition 4. (Time inconsistency) a) At $t = 1$, the competitive economy is trivially efficient — the efficiency proof is identical to the one used in proposition 2. Given the past choice of k_0, the following Euler equation for k_1 fully characterizes the equilibrium:

$$F' \left(k_0 - k_1\right) = \frac{\lambda_2}{\lambda_1 - \nu_1 \phi_1} G_2' \left(k_1\right) \tag{17}$$

Since this expression is different from equation (15) above when $\Omega \neq 0$, this shows that the planner would choose a different allocation when re-optimizing. This generates the time inconsistency.

b) The time consistent policy internalizes the choice of k_1 as a function of k_0, by imposing equation (17) as a constraint. This constraint restricts k_1 to be a given function of k_0, which I defined as $\tilde{k_1}(k_0)$. Imposing this constraint, assuming that experts' non-negativity constraints do not bind and imposing again that $\theta = \xi_0$ (as in the prove of proposition 2), the planner solves:

$$\begin{aligned}\mathcal{L} = \xi_0 c_0 + \xi_1 c_1 + \xi_2 c_2 - \lambda_0 (c_0 + p_0 k_0 + a_0 - w_0 + T_0) - \eta_0 a_0 + \nu_0 (a_0 + \phi_0 p_0 k_0) \\ -\lambda_1 \left(c_1 + F'\left(k_0 - \tilde{k_1}(k_0)\right)\left(\tilde{k_1}(k_0) - k_0\right) + a_1 - w_1 - G_1(k_0) - Ra_0\right) \\ -\eta_1 a_1 + \nu_1 \left(a_1 + \phi_1 F'\left(k_0 - \tilde{k_1}(k_0)\right)\tilde{k_1}(k_0)\right) \\ -\lambda_2 (c_2 - w_2 - G_2(k_1(k_0)) - Ra_1) \\ +\theta\left[T_0 + F(k_0 - k_1(k_0)) - F'\left(k_0 - \tilde{k_1}(k_0)\right)\left(k_0 - \tilde{k_1}(k_0)\right)\right]\end{aligned}$$

The Euler equation for the time consistent choice of k_0 is given by:

$$k_0 : p_0 = \frac{\lambda_1}{\lambda_0}\left(G_1'\left(k_0^{PT}\right) + F'\left(k_0^{PT} - k_1^{PT}\right)\right) + \frac{\nu_0}{\lambda_0}\phi_0 p_0 + \underbrace{\left(1 - \frac{d\tilde{k_1}(k_0)}{k_0}\right)}_{<1}\Omega$$

Note that the envelope theorem greatly simplifies the time consistent solution, by using the optimality conditions on k_0 and k_1. From equation (17), using the implicit function theorem, we can find that:

$$\frac{d\tilde{k_1}(k_0)}{k_0} = \frac{1}{1 + \alpha \frac{G_2''(k_1)}{F''(k_0 - k_1)}} < 1$$

Intuitively, by changing k_0 experts will have to sell less units of capital. The time consistent policy is then fully characterized by:

$$p_0\left(\frac{\xi_0 - \nu_0\phi_0}{\xi_1}\right) = G_1'\left(k_0^{PT}\right) + \frac{\xi_2}{\xi_1(1-\phi_1) + \xi_2\phi_1}G_2'(k_1) + \left(1 - \frac{d\tilde{k_1}(k_0)}{k_0}\right)\Omega$$

$$F'(k_0 - k_1) = \frac{\lambda_2}{\lambda_1 - \nu_1\phi_1}G_2'(k_1)$$

By comparing this characterization with equations (14) and (15), it can be easily shown that, in general, the time-consistent choice of k_0 may be higher or lower than the solution of the constrained planner.

Proofs: Section 5

Proposition 5. (Planner's solution with risk averse agents and uncertainty)

The Lagrangian corresponding to the planner's problem, after imposing the equilibrium prices pinned down by the risk neutral financiers, is now given by:

$$
\begin{aligned}
\mathcal{L} = U(c_0) + \mathbb{E}\left[U_s(c_{1s}) + U_s(c_{2s})\right] - \lambda_0^e\left(c_0 + p_0 k_0 + \mathbb{E}\left[a_{0s}\right] - w_0 + T_0\right) \\
-\mathbb{E}\left[\eta_{0s} a_{0s}\right] + \mathbb{E}\left[\nu_{0s}\left(a_{0s} + \phi_{0s} p_0 k_0\right)\right] \\
-\mathbb{E}\left[\lambda_{1s}^e\left(c_{1s} + p_{1s}\left(k_{1s} - k_0\right) + a_{1s} - w_{1s} - G_{1s}(k_0) - a_{0s}\right)\right] \\
-\mathbb{E}\left[\eta_{1s} a_{1s}\right] + \mathbb{E}\left[\nu_{1s}\left(a_{1s} + \phi_1 p_{1s} k_{1s}\right)\right] \\
-\mathbb{E}\left[\lambda_{2s}\left(c_{2s} - w_{2s} - G_{2s}(k_1) - a_{1s}\right)\right] \\
+\theta\left[\tilde{U}\left(c_0^f\right) + \mathbb{E}\left[\tilde{U}_s\left(c_{1s}^f\right) + \tilde{U}_s\left(c_{2s}^f\right)\right] - \lambda_0^f\left(c_0^f - w_0^f - T_0\right)\right] \\
+\theta\left[-\mathbb{E}\left[\lambda_{1s}^f\left(c_{1s}^f - w_{1s}^f - F_s(k_0 - k_1) + p_{1s}(k_0 - k_1)\right)\right] - \mathbb{E}\left[\lambda_{2s}^f\left(c_{2s}^f - w_{2s}^f\right)\right]\right],
\end{aligned}
$$

where $p_{1s} = F'(k_0 - k_{1s})\,\mathbb{I}\left[k_0 - k_{1s} > 0\right] + \overline{p}\cdot\mathbb{I}\left[k_0 - k_{1s} \leq 0\right]$. Note that I have removed non-negativity constraints, since they will not bind given the assumed Inada conditions for utility. Note also that all expectations operators stand for $\mathbb{E}[x] = \sum_s \pi_s x_s$, where π_s define probabilities.

The optimality conditions are:

$$
c_{ts}, c_{ts}^f : \lambda_{ts}^e = U_s'(c_{ts}),\ \lambda_{ts}^f = \tilde{U}_s'\left(c_{ts}^f\right),\ t = 0,1,2;\ s = 1,\ldots,S
$$

$$
a_{0s} : \lambda_0^e = \lambda_{1s}^e + \nu_{0s} - \eta_{0s} \text{ and } a_{1s} : \lambda_{1s}^e = \lambda_{2s}^e + \nu_{0s} - \eta_{0s}
$$

$$
\begin{aligned}
k_0 :\ p_0\lambda_0^e = \mathbb{E}\left[\lambda_{1s}^e\left(G_{1s}'\left(k_0^P\right) + p_{1s}\right) + \nu_{0s}^e\phi_{0s} p_0\right] \\
+\mathbb{E}\left[\left(\lambda_{1s}^e - \theta\lambda_{1s}^f\right)\left(k_0^P - k_{1s}^P\right)F_s''(\cdot)\,\alpha_s + \nu_{1s}^e\phi_{1s}k_{1s}^P F_s''(\cdot)\,\alpha_s\right]
\end{aligned}
$$

$$
\begin{aligned}
k_{1s} :\ p_{1s}\lambda_{1s}^e = \lambda_{2s}^e G_{2s}'\left(k_1^P\right) + \nu_{1s}^e\phi_{1s} p_{1s} \\
-\mathbb{E}\left[\left(\lambda_{1s}^e - \theta\lambda_{1s}^f\right)\left(k_0^P - k_{1s}^P\right)F_s''(\cdot)\,\alpha_s + \nu_{1s}^e\phi_{1s}k_{1s} F_s''(\cdot)\,\alpha_s\right],\ s = 1,\ldots,S
\end{aligned}
$$

where $\alpha_s = \mathbb{I}_s\left[k_0^P - k_{1s}^P > 0\right]$ and the condition that allows to substitute for θ:

$$
-\lambda_0^e + \theta\lambda_0^f = 0 \Rightarrow \theta = \frac{\lambda_0^e}{\lambda_0^f} \tag{18}
$$

By substituting equation (18) into the Euler equation for k_0, we recover equation (9) in the paper.

Proposition 6. (Planner's solution with price dependent margins)

The problem solved by constrained planner is identical to the one stated in equation (13), but for the exchange of parameters ϕ_t with the functions $\Phi_t(\cdot)$. Using identical procedures as in previous sections, we can derive the following Euler equations for the constrained planner:

$$
\begin{aligned}
k_0 : & \lambda_0 p_0 = \lambda_1 \left(G_1' \left(k_0^P \right) + F' \left(k_0^P - k_1^P \right) \right) + \nu_0 \Phi_0 \left(\cdot \right) p_0 \\
& \quad + \left(\lambda_1 - \theta \right) \left(k_0^P - k_1^P \right) F'' \left(k_0^P - k_1^P \right) + \nu_1 \left(\Phi_1 + \Phi_1' \left(\cdot \right) F' \left(k_0^P - k_1^P \right) \right) k_1^P F'' \left(k_0^P - k_1^P \right) \\
k_1 : & F' \left(k_0^P - k_1^P \right) \lambda_1 = \lambda_2 G_2' \left(k_1^P \right) + \nu_1 \phi_1 F' \left(k_0^P - k_1^P \right) \\
& \quad - \left[\left(\lambda_1 - \theta \right) \left(k_0^P - k_1^P \right) F'' \left(k_0^P - k_1^P \right) + \nu_1 \left(\phi_1 + \Phi_1' \left(\cdot \right) F' \left(k_0^P - k_1^P \right) \right) k_1^P F'' \left(k_0^P - k_1^P \right) \right]
\end{aligned}
$$

Note that, when optimizing with respect to k_0 or k_1, a new term corresponding to $\Phi' \left(\cdot \right)$ emerges — this is the only difference with respect to the baseline model.

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